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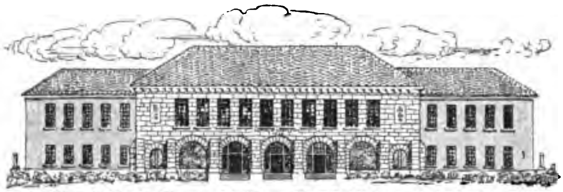

MATHEMATICAL TEXTS

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ADVANCED ALGEBRA

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PREFACE

This book is designed for use in secondary schools and in short college courses. It aims to present in concise but clear form the portions of algebra that are required for entrance to the most exacting colleges and technical schools.

The chapters on algebra to quadratics are intended for a review of the subject, and contain many points of view that should be presented to a student after he has taken a first course on those topics. Throughout the book the attention is concentrated on subjects that are most vital, pedagogically and practically, while topics that demand a knowledge of the calculus for their complete comprehension (as multiple roots, and Sturm's theorem) or are more closely related to other portions of mathematics (as theory of numbers, and series) have been omitted.

The chapter on graphical representation has been introduced early, in the belief that the illumination which it affords greatly enlivens the entire presentation of algebra. The discussion of the relation between pairs of linear equations and pairs of straight lines is particularly suggestive.

In each chapter the discussion is directed toward a definite result. The chapter on theory of equations aims to give a simple and clear treatment of the method of obtaining the real roots of an equation and the theorems that lead to that

process. Similarly direct in its argument is the chapter on determinants, its object being the solution of non-homogeneous equations and the necessary evaluation of determinants.

I am under obligations to many friends and colleagues for suggestions, but especially to Professor P. F. Smith, who has read the book both in manuscript and proof and whose numerous suggestions have been invaluable.

New Haven, Connecticut
August, 1905

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ADVANCED ALGEBRA

ALGEBRA TO QUADRATICS

CHAPTER I

FUNDAMENTAL OPERATIONS

1. It is assumed that the elementary operations and the meaning of the usual symbols of algebra are familiar and do not demand detailed treatment. In the following brief exposition of the formal laws of algebra most of the proofs are omitted.

2. Addition. The process of adding two positive integers a and b consists in finding a number x such that

$$a + b = x.$$

For any two given positive integers a single **sum** x exists which is itself a positive integer.

3. Subtraction. The process of subtracting the positive number b from the positive number a consists in finding a number x such that

$$b + x = a. \quad (1)$$

This number x is called the **difference** between a and b and is denoted as follows:

$$a - b = x,$$

a being called the **minuend** and b the **subtrahend.**

If $a > b$ and both are positive integers, then a single positive integer x exists which satisfies the condition expressed by equation (1)

If $a < b$, then x is not a positive integer. In order that the process of subtraction may be possible in this case also, we introduce negative numbers which we symbolize by $(-a)$, $(-b)$, etc. When in the difference $a-b$, a is less than b, we define $a-b=(-(b-a))$. The processes of addition and subtraction for the negative numbers are defined as follows:

$$(-a)+(-b)=(-(a+b)).$$
$$(-a)+b=(-(a-b)).$$
$$a+(-b)=a-b.$$
$$(-a)-(-b)=(-(a-b)).$$
$$(-a)-b=(-(a+b)).$$
$$a-(-b)=a+b.$$
$$(-(-a))=a.\text{*}$$

4. Zero. If in equation (1), $a=b$, there is no positive or negative number which satisfies the equation. In order that in this case also the equation may have a number satisfying it, we introduce the number zero which is symbolized by 0 and defined by the equation

$$a+0=a,$$

or

$$a-a=0.$$

The processes of addition and subtraction for this new number zero are defined as follows, where α stands for either a positive or a negative number

$$0+\alpha=\alpha\pm 0=\alpha.$$
$$0-\alpha=-\alpha.$$
$$0\pm 0=0.$$

5. Multiplication. The process of multiplying a by b consists in finding a number x which satisfies the equation

$$a\cdot b=x.$$

* The symbol for a positive integer might be written $(+a)$, $(+b)$, etc., consistently with the notation for negative numbers. Since, however, no ambiguity results, we omit the + sign. Since the laws of combining the + and − signs given in this and the following paragraphs remove the necessity for the parentheses in the notation for the negative number, we shall omit them where no ambiguity results.

When a and b are positive integers x is a positive integer which may be found by adding a to itself b times. When the numbers to be multiplied are negative we have the following laws,

$$(-a)\cdot(-b)=a\cdot b,$$
$$(-a)\cdot b=a\cdot(-b)=-(a\cdot b),$$
$$0\cdot\alpha=\alpha\cdot 0=0, \tag{1}$$

where α is a positive or negative number or zero.

These symbolical statements include the statement of the following

PRINCIPLE. *A product of numbers is zero when and only when one or more of the factors are zero.*

This most important fact, which we shall use continually, assures us that when we have a product of several numbers as

$$a\cdot b\cdot c\cdot d=e,$$

first, if e equals zero, it is certain that one or more of the numbers a, b, c, or d are zero; *second*, if one or more of the numbers a, b, c, or d are zero, then e is also zero.

6. Division. The process of dividing α by β consists in finding a number x which satisfies the equation

$$x\cdot\beta=\alpha, \tag{1}$$

where α and β are positive or negative integers, or α is 0.

When α occurs in the sequence of numbers

$$\cdots -3\beta,\ -2\beta,\ -\beta,\ 0,\ \beta,\ 2\beta,\ 3\beta,\ \cdots,$$

x is a definite integer or 0, that is, it is a number such as we have previously considered. If α is not found in this series, but is between two numbers of the series, then in order that in this case the process may also be possible we introduce the fraction which we symbolize by $\alpha\div\beta$ or $\dfrac{\alpha}{\beta}$ and which is defined by the equation

$$\frac{\alpha}{\beta}\cdot\beta=\alpha.$$

The operations for addition, subtraction, multiplication, and division of fractions are defined as follows:

$$\frac{\alpha}{\beta} \pm \frac{\gamma}{\delta} = \frac{\alpha\delta \pm \beta\gamma}{\beta\delta}, \tag{2}$$

$$\frac{\alpha}{\beta} \cdot \frac{\gamma}{\delta} = \frac{\alpha\gamma}{\beta\delta}, \tag{3}$$

$$\frac{\alpha}{\beta} \div \frac{\gamma}{\delta} = \frac{\alpha\delta}{\beta\gamma}. \tag{4}$$

Further properties of fractions are the following:

$$1 = \frac{\alpha}{\alpha} = \frac{1}{1},$$

$$\frac{\alpha}{\beta} = \frac{\delta\alpha}{\delta\beta}, \text{ where } \delta \text{ is any number,} \tag{5}$$

$$\frac{-\alpha}{\beta} = \frac{\alpha}{-\beta} = -\frac{\alpha}{\beta}. \tag{6}$$

The last two equations are expressed verbally as follows:

Both numerator and denominator of a fraction may be multiplied by any number without changing the value of the fraction.

Changing the sign of either numerator or denominator of a fraction is equivalent to changing the sign of the fraction.

The laws of signs in multiplication given on p. 3 may now be assumed to hold when the letters represent fractions as well as integers.

Thus for example $\left[-\left(\frac{a}{b}\right)\right] \cdot \left[-\left(\frac{c}{d}\right)\right] = \frac{ac}{bd}.$

The positive or negative number α may be written in the fractional form

$$\frac{\alpha}{1}.$$

7. Division by zero. If in equation (1), § 6, $\beta = 0$, there is no single number x which satisfies the equation, since by (1), § 5, whatever value x might have, its product with zero would be zero.

Thus division by zero is entirely excluded from algebraic processes. Before a division can safely be performed one must be assured that the divisor cannot vanish. In the equation

$$4 \cdot 0 = 2 \cdot 0,$$

if we should allow division of both sides of our equation by zero, we should be led to the absurd result $4 = 2$.

8. Fundamental operations. The operations of addition, subtraction, multiplication, and division we call the four **fundamental operations.** Any numbers that can be derived from the number 1 by means of the four fundamental operations we call **rational numbers.** Such numbers comprise all positive and negative integers and fractions. Positive or negative integers are called **integral numbers.**

9. Practical demand for negative and fractional numbers. In the preceding discussion negative numbers and fractions have been introduced on account of the *mathematical* necessity for them. They were needed to make the four fundamental operations always possible. That this mathematical necessity corresponds to a practical necessity appears as soon as we attempt to apply our four operations to practical affairs. Thus if on a certain day the temperature is $+20°$ and the next day the mercury falls 25°, in order to express the second temperature we must subtract 25 from 20. If we had not introduced negative numbers, this would be impossible and our mathematics would be inapplicable to this and countless other everyday occurrences.

10. Laws of operation. All the numbers which we use in algebra are subject to the following laws.

Commutative law of addition. This law asserts that the value of the sum of two numbers does not depend on the order of summation.

Symbolically expressed,

$$a + b = b + a,$$

where a and b represent any numbers such as we have presented or shall hereafter introduce.

Associative law of addition. This law asserts that the sum of three numbers does not depend on the way in which the numbers are grouped in performing the process of addition.

Symbolically expressed,

$$a + (b + c) = (a + b) + c = a + b + c.$$

Commutative law of multiplication. This law asserts that the value of the product of two numbers does not depend on the order of multiplication.

Symbolically expressed,

$$a \cdot b = b \cdot a.$$

Associative law of multiplication. This law asserts that the value of the product of three numbers does not depend on the way in which the numbers are grouped in the process of multiplication.

Symbolically expressed,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c.$$

Distributive law. This law asserts that the product of a single number and the sum of two numbers is identical with the sum of the products of the first number and the other two numbers taken singly.

Symbolically expressed,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

All the above laws are readily seen to hold when more than three numbers are involved.

11. Integral and rational expressions. A polynomial is **integral** when it may be expressed by a succession of literal terms, no one of which contains any letter in the denominator.

Thus $4x^5 - x^3 - 2x^2 - \frac{3}{4}x + 1$ is integral.

The quotient of two integral expressions is called **rational.**

Thus $\dfrac{x^2 - 2x + 3}{x - 7}$ is rational.

12. Operations on polynomials. We assume that the same formal laws for the four fundamental operations enunciated in §§ 2–6 and the laws given in § 10 hold whether the letters in the symbolic statements represent numbers or polynomials.

In fact the literal expressions which we use are in essence nothing else than numerical expressions, since the letters are merely symbols for numbers. When the letters are replaced by numbers, the literal expressions reduce to numerical expressions for which the previous laws have been explicitly given.

13. Addition of polynomials. For performing this operation we have the following

RULE. *Write the terms with the same literal part in a column. Find the algebraic sum of the terms in each column, and write the results in succession with their proper signs.*

When the polynomials reduce to monomials the same rule is to be observed.

EXERCISES

Add the following:

1. $3a^2b^2 - 2ab + 6a^2b - a$; $4ab - 2a^2b^2 - 11a^2b + 9a$; $2a^2b - ab - 2a$; $8a^2b^2 - 4ab - 6a$.

Solution:

$$\begin{array}{rrrr} 3a^2b^2 & -\ 2ab & +\ 6a^2b & -\ a \\ -\ 2a^2b^2 & +\ 4ab & -\ 11a^2b & +\ 9a \\ & -\ ab & +\ 2a^2b & -\ 2a \\ \underline{8a^2b^2} & \underline{-\ 4ab} & & \underline{-\ 6a} \\ 9a^2b^2 & -\ 3ab & -\ 3a^2b & \end{array}$$

2. $21a - 24b - 8c^2$; $16c + 17b + 6c^2 - 20a$; $18b - 18c$.
3. $x^4 - 6x^2 - 8x - 1$; $2x^3 + 1$; $6x^2 + 7x + 2$; $x^4 - x^3 + x - 1$.
4. $9(a + b) - 6(b + c) + 7(a + c)$; $4(b + c) - 7(a + b) - 8(a + c)$; $(a + c) - (a + b) + (b + c)$.
5. $a^2 - 4ab + b^2 + a + b - 2$; $2a^2 + 4ab - 3b^2 - 2a - 2b + 4$; $3a^2 - 5ab - 4b^2 + 3a + 3b - 2$; $6a^2 + 10ab + 5b^2 + a + b$.

14. Subtraction of polynomials. For performing this operation we have the following

RULE. *Write the subtrahend under the minuend so that terms with the same literal part are in the same column.*

To each term of the minuend add the corresponding term of the subtrahend, the sign of the latter having been changed.

It is generally preferable to imagine the signs of the subtrahend changed rather than actually to write it with the changed signs.

EXERCISES

1. From $a^2b^2 - 3a^2b + 8ab + 6b$ subtract $9a^2b^2 - 6ab + 4a^2b + a$.

Solution:

$$
\begin{array}{r}
a^2b^2 - 3a^2b + 8ab + 6b\phantom{{}-a} \\
9a^2b^2 + 4a^2b - 6ab \phantom{{}+6b} + a \\
\hline
-8a^2b^2 - 7a^2b + 14ab + 6b - a
\end{array}
$$

2. From $6abx - 4mn + 5x$ subtract $3mn + 6ax - 4abx$.

3. From $m + an + bq$ subtract the sum of

$$cm + dn + (b - a)q \text{ and } (a - b)q - (a + d)n - cm.$$

4. From the sum of $\frac{4}{3}a + \frac{9}{10}b + \frac{1}{2}c$ and $-b - c - a$ subtract $\frac{1}{2}b - \frac{3}{4}c + \frac{1}{9}a$.

5. From the sum of $2x^2 - 3x + 4$ and $x^4 - \frac{3}{4}x - \frac{1}{2}$ subtract $x^3 - \frac{3}{2}x^2 - 3\frac{3}{4}x + 3\frac{1}{2}$.

15. Parentheses. When it is desirable to consider as a single symbol an expression involving several numbers or symbols for numbers, the expression is inclosed in a parenthesis. This parenthesis may then be used in operations as if it were a single number or symbol, as in fact it is, excepting that the operations inside the parenthesis may not yet have been carried out.

RULE. *When a single parenthesis is preceded by a + sign the parenthesis may be removed, the various terms retaining the same sign.*

When a single parenthesis is preceded by a − sign the parenthesis may be removed, providing we change the signs of all the terms inside the parenthesis.

When several parentheses occur in an expression we have the following

RULE. *Remove the innermost parenthesis, changing the signs of the terms inside if the sign preceding it is minus.*

Simplify, if possible, the expression inside the new innermost parenthesis.

Repeat the process until all the parentheses are removed.

It is in general unwise to shorten the process by carrying out some of the steps in one's head. The liability to error in such attempts more than offsets the gain in time.

EXERCISES

Remove parentheses from the following:

1. $b - \{9a - [2b + (4a - \overline{2a - b}) - 6b]\}$.

Solution:

$$\begin{aligned} & b - \{9a - [2b + (4a - \overline{2a - b}) - 6b]\} \\ &= b - \{9a - [2b + (4a - 2a + b) - 6b]\} \\ &= b - \{9a - [2b + (2a + b) - 6b]\} \\ &= b - [9a - (2b + 2a + b - 6b)] \\ &= b - [9a - (2a - 3b)] \\ &= b - (9a - 2a + 3b) \\ &= b - (7a + 3b) \\ &= b - 7a - 3b \\ &= -7a - 2b. \end{aligned}$$

2. $-\{-[-[-(-(-1))]]\}$.

3. $a^2 + 4 - \{6 - [-(a^2 - b) + 1]\}$.

4. $\frac{1}{3}\{\frac{1}{4} - \frac{2}{3}[\frac{3}{4} - \frac{1}{2}(\frac{1}{6} - \frac{1}{8} \cdot \frac{2}{3} - \frac{5}{12}) + \frac{4}{3}] - \frac{1}{2}\}$.

5. $x^2 - \{y^2 - [4x + 3(y - 9x(y - x)) + 9y(x - y)]\}$.

6. Find the value of $a - \{5b - [a - (3c - 3b) + 2c - 3(a - \overline{2b - c})]\}$ when $a = -3$, $b = 4$, $c = -5$.

16. Multiplication. It is customary to write $a \cdot a = a^2$; $a \cdot a \cdot a = a^3$; $\underbrace{a \cdot a \cdots a}_{n \text{ terms}} = a^n$. We have then by the associative law of multiplication, § 10,

$$a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5,$$

or, in general,

$$a^m a^n = a^{m+n}, \tag{I}$$

where m and n are positive integers.

Furthermore,

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdots a^n}_{m \text{ terms}} = a^{n \cdot m}. \tag{II}$$

Finally,

$$a^n \cdot b^n = (a \cdot b)^n. \tag{III}$$

The distinction between $(a^n)^m$ and a^{n^m} should be noted carefully. Thus $(2^3)^2 = 8^2 = 64$, while $2^{3^2} = 2^9 = 512$.

Equation (I) asserts that the exponent in the product of two powers of any expression is the sum of the exponents of the factors. Hence we may multiply monomials as follows:

RULE. *Write the product of the numerical coefficients, followed by all the letters that occur in the multiplier and multiplicand, each having as its exponent the sum of the exponents of that letter in the multiplier and multiplicand.*

EXAMPLE. $4\,a^2b^{10}cd^4 \cdot (-16\,a^4bd^7) = -64\,a^6b^{11}cd^{11}$.

17. Multiplication of monomials by polynomials. By the distributive law, § 10, we can immediately formulate the

RULE. *Multiply each term of the polynomial by the monomial and write the resulting terms in succession.*

EXAMPLE.

$$\begin{array}{r} 9\,a^2b^2 - 2\,ab + 4\,ab^2 - a + b^6 \\ 3\,a^2b \\ \hline 27\,a^4b^3 - 6\,a^3b^2 + 12\,a^3b^3 - 3\,a^3b + 3\,a^2b^7 \end{array}$$

18. Multiplication of polynomials. If in the expression for the distributive law,

$$a(c + d) = ac + ad,$$

we replace a by $a + b$, we have

$$(a + b)(c + d) = ac + bc + ad + bd,$$

which affords the

RULE. *Multiply the multiplicand by each term of the multiplier in turn, and write the partial products in succession.*

To test the accuracy of the result assume some convenient numerical value for each letter, and find the corresponding numerical value of multiplier, multiplicand, and product. The latter should be the product of the two former.

EXERCISES

1. Multiply and check the following:

(*a*) $2\,a^2 + ab + 4\,b^2 + b$ and $a - b + ab$.

Solution:

$$\begin{array}{llllllll} 2\,a^2 + & ab & + 4\,b^2 & + \; b & & & & \\ & a & - \; b & + \; ab & & & & \\ \hline 2\,a^3 + & a^2b & + 4\,ab^2 & + \; ab & & & & \\ & - 2\,a^2b & - \; ab^2 & & - 4\,b^3 - b^2 & & & \\ & & + \; ab^2 & & & + 2\,a^3b + a^2b^2 + 4\,ab^3 & & \\ \hline 2\,a^3 - & a^2b & + 4\,ab^2 & + \; ab & - 4\,b^3 - b^2 & + 2\,a^3b + a^2b^2 + 4\,ab^3 & = 8 & \end{array}$$

Check:

Let $a = b = 1$

$= 8$

$= 1$

(b) $6abx^2$ and $4a^2b^3x$.

(c) $\dfrac{x^2y^3z^4}{3}$ and $-9x^3y^2z$.

(d) $3ab^2x - \frac{1}{2}bx^4$ and $6a^3cx$.

(e) $x^{2a} + y^{2b} + x^ay^b$ and $x^a - y^b$.

(f) $a^2 + ab + b^2$ and $a^2 + ac + c^2$.

(g) x^ny^{m-2}, $x^{n-3}y^{m+4}$, and x^3y^{2m-2}.

(h) $x^{p-3} + x^{p-2} + 1$ and $x^3 - x^2 - 1$.

(i) $8a^2bc$, $\dfrac{3}{4}ab^2$, $-7b^2$, $-\dfrac{3}{14}a^3c^4$, and $\dfrac{c}{6}$.

(j) $ax^4 - 2a^2x^3 - x + 4a$ and $-x + 2a$.

(k) $x^{a+b} + x^{2a} + x^{2b} + x^{3a-b}$ and $x^{a-b} - 1$.

(l) $15x^4 - 11x^3 + 6x^2 + 2x - 1$ and $-3x^2 - 1$.

(m) $4x^4 - 8xy^3 + \frac{1}{6}x^2y^2 - \frac{3}{7}x^2y - x - y$ and $-42xy$.

2. Expand $(x + y)^4$.

3. Expand and simplify

$$(x^2 + y^2 + z^2)^2 - (x + y + z)(x + y - z)(x + z - y)(y + z - x).$$

19. Types of multiplication. The following types of multiplication should be so familiar as merely to require inspection of the factors in order to write the product.

RULE. *The product of the sum and difference of two terms is equal to the square of the terms with like signs minus the square of the terms which have unlike signs.*

EXAMPLES.
$$(a - b)(a + b) = a^2 - b^2.$$
$$(4x^2 - 3y^2)(4x^2 + 3y^2) = 16x^4 - 9y^4.$$

20. The square of a binomial. This process is performed as follows:

RULE. *The square of a binomial, or expression in two terms, is equal to the sum of the squares of the two terms plus twice their product.*

EXAMPLES.
$$(x + y)^2 = x^2 + y^2 + 2xy.$$
$$(2a - 3b)^2 = 4a^2 + 9b^2 - 12ab.$$

21. The square of a polynomial. This process is performed as follows:

RULE. *The square of any polynomial is equal to the sum of the squares of the terms plus twice the product of each term by each term that follows it in the polynomial.*

EXAMPLE. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

22. The cube of a binomial. This process is performed as follows:

RULE. *The cube of any binomial is given by the following expression:*

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

EXERCISES

Perform the following processes by inspection.

1. $(a-b+c)^2$.
2. $(a^4-b^6)^2$.
3. $(2x^{r-1}-1)^2$.
4. $(a^2-b^2)^3$.
5. $(2x^{r-1}-1)^3$.
6. $(1-8x^2y)^2$.
7. $(x^2-2x+1)^2$.
8. $(x^2-y^2+z^2)^2$.
9. $(2a-2b-c)^2$.
10. $(x^3-2x-1)^2$.
11. $(a^{p-3}-b^{p+3})^2$.
12. $(-6x^2y+4xy^2)^3$.
13. $(x^p-y^q)(x^p+y^q)$.
14. $(-6x^2y+4xy^2)^2$.
15. $(3x+2y)(3x-2y)$.
16. $(-3ax^2+2ax-b)^2$.
17. $(-3x^2y+\frac{1}{2}z^2)(3x^2y+\frac{1}{2}z^2)$.
18. $(-4-6a^2b)(-4+6a^2b)$.
19. $\left(2a-\frac{b}{2}\right)^2$.
20. $\left(2a-\frac{b}{3}\right)^3$.

23. Division. By the definition of division in § 6, we have

$$a = \frac{a^2}{a},\quad a^2 = \frac{a^3}{a},\quad a^3 = \frac{a^5}{a^2};$$

or, in general,

$$a^{n-m} = \frac{a^n}{a^m}, \tag{I}$$

where n and m are positive integers and $n > m$.

If $n = m$, we preserve the same principle and write

$$a^{n-n} = a^0 = \frac{a^n}{a^n} = 1.$$

24. Division of monomials. Keeping in mind the rule of signs for division given in § 6, we have the following

RULE. *Divide the numerical coefficient of the dividend by that of the divisor for the numerical coefficient of the quotient, keeping in mind the rule of signs for division.*

Write the literal part of the dividend over that of the divisor in the form of a fraction, and perform on each pair of letters occurring in both numerator and denominator the process of division as defined by equation (I) *in the preceding paragraph.*

EXAMPLE. Divide $12\,a^4b^{11}c^2d$ by $-6\,a^9bc^2d^8$.

$$\frac{12\,a^4b^{11}c^2d}{-6\,a^9bc^2d^8} = -\frac{2\,b^{10}}{a^5d^7}.$$

25. Division of a polynomial by a monomial. This process is performed as follows:

RULE. *Divide each term of the polynomial by the monomial and write the partial quotients in succession.*

EXAMPLE. Divide $8\,a^2b^6 - 12\,a^6b^2$ by $2\,a^3b^3$.

$$\frac{8\,a^2b^6}{2\,a^3b^3} - \frac{12\,a^6b^2}{2\,a^3b^3} = \frac{4\,b^3}{a} - \frac{6\,a^3}{b}.$$

26. Division of a polynomial by a polynomial. This process is performed as follows:

RULE. *Arrange both dividend and divisor in descending powers of some common letter (called the letter of arrangement).*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the divisor by this first term of the quotient and subtract the product from the dividend.

Divide the first term of this remainder by the first term of the divisor for the second term of the quotient, and proceed as before until the remainder vanishes or is of lower degree in the letter of arrangement than the divisor.

When the last remainder vanishes the dividend is exactly divisible by the divisor. This fact may be expressed as follows:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}.$$

When the last remainder does not vanish we may express the result of division thus:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

The coefficients in the quotient will be rational numbers if those in both dividend and divisor are rational.

EXERCISES

Divide and check the following:

1. $8a^3 + 6a^2b + 9ab^2 + 9b^3$ by $4a + b$.

Solution:

$$\begin{array}{r|l|l}
4a+b & 8a^3 + 6a^2b + 9ab^2 + 9b^3 & 2a^2 + ab + 2b^2 \\
 & \underline{8a^3 + 2a^2b} & \\
 & \quad 4a^2b + 9ab^2 & \\
 & \quad \underline{4a^2b + ab^2} & \\
 & \qquad 8ab^2 + 9b^3 & \\
 & \qquad \underline{8ab^2 + 2b^3} & \\
 & \qquad\qquad 7b^3 &
\end{array}$$

Result: $2a^2 + ab + 2b^2 + \dfrac{7b^3}{4a + b}$.

Check: Let $a = b = 1$. Dividend $= 32$, divisor $= 5$, quotient $= 6\frac{2}{5}$.

$$32 \div 5 = 6\tfrac{2}{5}.$$

2. $x^{12} - y^{12}$ by $x^3 - y^3$.

3. $2x^2 - 5x + 2$ by $x - 2$.

4. $x^{12} - y^{12}$ by $x^4 - y^4$.

5. $x^6 - y^6$ by $x^2 + xy + y^2$.

6. $a^3 - a^2 + 2$ by $a + 1$.

7. $-63x^4y^3z^2$ by $-9x^2y^3z$.

8. $x^2 - x - 30$ by $x + 5$.

9. $4a^2b - 6ab^2 - 2a$ by $-2a$.

10. $16a^2b^4c^{11}$ by $-2a^2b^3c^7$.

11. $\frac{1}{3}x^2 - 3\frac{1}{4}x - \frac{5}{6}$ by $1\frac{1}{5}x + \frac{3}{10}$.

12. $ax^2 + (a^2 - b)x - ab$ by $x + a$.

13. $ax^n + bx^{n-1} + cx^{n-2} - dx^{n-3}$ by x^3.

14. $16a^2x^2y^2 - 8ax^3y^2 - 4x^4y$ by $-\frac{4}{3}xy$.

15. $18a^pb^q + 6a^{p+2}b^{q+3} - 9a^{p+1}b^q$ by $3a^pb^q$.

16. $a^2 - 2ab - 4c^2 + 8bc - 3b^2$ by $a - 2c + b$.

17. $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$ by $x - a$.

18. $2y^2 - 6x^2 + \frac{13}{5}xy + \frac{19}{6}x - \frac{257}{60}y + 1$ by $2x + \frac{4}{5}y - \frac{3}{2}$.

19. $x^3 - 2x^2y - x^2 + y^2x + 2xy - y - y^2 + 2$ by $x - y + 1$.

20. $3x^3 + 6x^2y + 9x^2 + 2xy^2 + 5y^3 + 2y + 6y^2 + 3$ by $x + 2y + 3$.

27. Types of division. The following types of division, which may be verified by the rule just given for any particular integral value of n, should be so familiar that they may be performed by inspection.

$$a^{2n} - b^{2n} \div a^n \pm b^n = a^n \mp b^n. \tag{I}$$

$$a^n + b^n \div a + b = a^{n-1} - a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}, \tag{II}$$

where n is odd.

$$a^n - b^n \div a - b = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}, \tag{III}$$

where n is odd or even.

EXERCISES

Give by inspection the results of the following divisions.

1. $a^6 - 1 \div a - 1.$
2. $a^5 + 1 \div a + 1.$
3. $x^7 + 128 \div x + 2.$
4. $x^5 + y^5 \div x + y.$
5. $x^6 - y^6 \div x^3 + y^3.$
6. $x^4 - y^4 \div x - y.$
7. $x^8 - y^8 \div x^4 - y^4.$
8. $a^{2m} - 1 \div a - 1.$
9. $a^{2n+1} - 1 \div a - 1.$
10. $27\,a^9 + 8\,b^3 \div 3\,a^3 + 2\,b.$
11. $8\,x^3 - 27 \div 2\,x - 3.$
12. $4\,a^2 - 256\,b^8 \div 2\,a + 16\,b^4.$
13. $16\,a^4 - 256 \div 4\,a^2 + 16.$
14. $27\,a^{12} - 64\,b^{12} \div 3\,a^4 - 4\,b^4.$

CHAPTER II

FACTORING

28. Statement of the problem. The operation of *division* consists in finding the quotient when the dividend and divisor are given. The product of the quotient and the divisor is the dividend, and the quotient and the divisor are the factors of the dividend. Thus the process of division consists in finding a second factor of a given expression when one factor is given.

The process of *factoring* consists in finding all the factors of a polynomial when no one of them is given. This operation is in essence the reverse of the operation of multiplication. We shall be concerned only with those factors that have rational coefficients.

29. Monomial factors. By the distributive law, § 10,

$$ab + ac = a(b + c).$$

This affords immediately the

RULE. *Write the largest monomial factor which occurs in every term outside a parenthesis which includes the algebraic sum of the remaining factors of the various terms.*

EXERCISES

Factor the following:

1. $6a^2b^9c + 9ab^6c^4 - 15a^4bc^7$.

Solution: $6a^2b^9c + 9ab^6c^4 - 15a^4bc^7 = 3abc(2ab^8 + 3b^5c^3 - 5a^3c^6)$.

2. $14anx - 21bnx - 7n$.

3. $121a^2b^3c - 22a^3bc^2 + 11ab^2c^3$.

4. $5x^4y^2 - 10x^3y^3 - 5x^2y^4 - 15x^2y^2$.

5. $21abn + 6ab^2n^2 - 18a^2bn^2 + 15a^2b^2n$.

6. $10ab^2cmx - 5ab^2cy + 5ab^2cz - 15abc^2m^2$.

7. $7a^2x^3y^4 - 49ax^3y^4 + 14axy^3z^2 - 21a^2x^2y^2$.

8. $45a^4b^2c^3d - 9ab^4c^2d^3 + 27a^3bc^4d^2 - 117a^2b^3cd^4$.

30. Factoring by grouping terms. If in the expression for the distributive law,
$$ac + bc = (a + b)c,$$
we replace c by $c + d$,

we have $a(c + d) + b(c + d) = ac + ad + bc + bd.$

We may then factor the right-hand member as follows:
$$ac + ad + bc + bd = a(c + d) + b(c + d) = (a + b)(c + d).$$

This affords the

RULE. *Factor out any monomial expression that is common to each term of the polynomial.*

Arrange the terms of the polynomial to be factored in groups of two or more terms each, such that in each group a monomial factor may be taken outside a parenthesis which in each case contains the same expression.

Write the algebraic sum of the monomial factors that occur outside the various parentheses for one factor, and the expression inside the parentheses for the other factor.

EXERCISES

Factor the following:

1. $4a^3b - 6a^2b^2 - 4a^4 + 6ab^3$.

Solution:
$$\begin{aligned} &4a^3b - 6a^2b^2 - 4a^4 + 6ab^3 \\ &= 2a(2a^2b - 3ab^2 - 2a^3 + 3b^3) \\ &= 2a(2a^2b - 2a^3 - 3ab^2 + 3b^3) \\ &= 2a[2a^2(b - a) + 3b^2(b - a)] \\ &= 2a(b - a)(2a^2 + 3b^2). \end{aligned}$$

2. $x^2 - (3a + 4b)x + 12ab$.

Solution:
$$\begin{aligned} &x^2 - (3a + 4b)x + 12ab \\ &= x^2 - 3ax - 4bx + 12ab \\ &= x(x - 3a) - 4b(x - 3a) \\ &= (x - 4b)(x - 3a). \end{aligned}$$

3. $2ax - 3by - 2bx + 3ay$.
4. $56a^2 - 40ab + 63ac - 45bc$.
5. $a^px^2 - b^qy^3 - b^qx^2 + a^py^3$.
6. $91x^2 - 112mx + 65nx - 80mn$.
7. $ax - bx + cx + ay - by + cy$.
8. $2ax - bx - cb + 2ab + 2ac - b^2$.

9. $2x^5 - 3x^3 + 2x^2 - 3.$

10. $x^3 + x^2 + x + 1.$

11. $acx^2 - bcx + adx - bd.$

12. $2bx - x^2 - 4b + 2x.$

13. $3x^3 - 12x^3y^2 - 4y^2 + 1.$

14. $4x^2 - (8 + b)x + 2b.$

15. $x^4 - (4m + 9n)x^2 + 36mn.$

16. $x^4 - (2m + 3n)x^2 + 6mn.$

17. $a^2x - ac + aby - ab^2x - b^3y + cb^2.$

18. $18a^3 - 2ac^4b^6 - 9a^2b + c^4b^7.$

19. $2ax - 5ay + a - 2bx + 5by - b.$

20. $2ax - ay - 2bx + 4cx - 2cy + by.$

21. $2a^2x^6 + 4a^2x^4 + 2a^2x^2 + 4a^2.$

22. $8x^2 - 2ax - 12xz + 3az + 4xy - ay.$

31. Factors of a quadratic trinomial. In this case we cannot factor by grouping terms immediately, as that method is inapplicable to a polynomial of less than four terms. We observe, however, that in the product of two binomial expressions,

$$(mx + n)(px + q) = mpx^2 + (mq + np)x + nq,$$

the coefficient of x is the sum of two expressions mq and np, whose product is equal to the product of the coefficient of x^2 and the last term, that is,

$$mq \cdot np = mp \cdot nq.$$

Thus, to factor the right-hand member of this equation, we may remove the parenthesis from the term in x and use the principle of grouping terms. Thus

$$\begin{aligned} mpx^2 &+ (mq + np)x + nq \\ &= mpx^2 + mqx + npx + nq \\ &= mx(px + q) + n(px + q) \\ &= (mx + n)(px + q). \end{aligned}$$

This affords the

RULE. *Write the trinomial in order of descending powers of x (or the letter in which the expression is quadratic).*

Multiply the coefficient of x^2 by the term not involving x, and find two factors of this product whose algebraic sum is the coefficient of x.

Replace the coefficient of x by this sum and factor by grouping terms.

Factoring a perfect square is evidently a special case under this method. Thus factor $x^2 + 6x + 9$.

$$1 \cdot 9 = 9. \qquad 3 + 3 = 6.$$

$$\begin{aligned} &x^2 + (3 + 3)x + 9 \\ &= x^2 + 3x + 3x + 9 \\ &= x(x + 3) + 3(x + 3) \\ &= (x + 3)(x + 3) \\ &= (x + 3)^2. \end{aligned}$$

One will usually recognize when a trinomial is a perfect square, in which case the factors may be written down by inspection.

EXERCISES

Factor the following:

1. $3x^2 + 8x + 4$.

Solution: $3 \cdot 4 = 12. \qquad 6 + 2 = 8.$

$$\begin{aligned} &3x^2 + 8x + 4 \\ &= 3x^2 + (6 + 2)x + 4 \\ &= 3x^2 + 6x + 2x + 4 \\ &= 3x(x + 2) + 2(x + 2) \\ &= (3x + 2)(x + 2). \end{aligned}$$

2. $8x^2 - 14bx + 3b^2$.

Solution: $8 \cdot 3b^2 = 24b^2. \qquad -2b - 12b = -14b.$

$$\begin{aligned} &8x^2 - 14bx + 3b^2 \\ &= 8x^2 - (2b + 12b)x + 3b^2 \\ &= 8x^2 - 12bx - 2bx + 3b^2 \\ &= 4x(2x - 3b) - b(2x - 3b) \\ &= (4x - b)(2x - 3b). \end{aligned}$$

3. $28x^2 - 3x - 40$.

Solution: $28 \cdot (-40) = -1120.$

The factors of 1120 must be factors of 28 and 40. We seek two factors of 1120, one of which exceeds the other by 3. We note that since 40 exceeds 28 by more than 3, one factor must be greater and the other less than 28 and 40 respectively.

Since $\qquad 4 \cdot 7 = 28$ and $5 \cdot 8 = 40$,

we try $\qquad 5 \cdot 7 = 35$ and $4 \cdot 8 = 32$,

which are the required factors of 1120.

$$\begin{aligned} &28x^2 - 3x - 40 \\ &= 28x^2 - (35 - 32)x - 40 \\ &= 28x^2 - 35x + 32x - 40 \\ &= 7x(4x - 5) + 8(4x - 5) \\ &= (7x + 8)(4x - 5). \end{aligned}$$

4. $x^2 - 6x + 9$.
5. $2x^2 + x - 6$.
6. $2x^2 - x - 6$.
7. $2x^2 + x - 91$.
8. $x^2 + x - 182$.
9. $9x^2 - 2x - 7$.
10. $2x^2 + 5x + 8$.
11. $x^2 - 11x + 18$.
12. $3x^2 - 10x - 8$.
13. $6x^2 + 17x + 7$.
14. $15x^2 + 4x - 3$.
15. $7y^2 - 4y - 11$.
16. $a^2 - 6ab + 9b^2$.
17. $5x^2 - 12x + 4$.
18. $18x^2 - 73x + 4$.
19. $27x^2 + 3x - 2$.
20. $24x^2 - 31x - 15$.
21. $21x^2 - 31x + 4$.
22. $12x^2 + 60x - 72$.
23. $x^4 - 3ax^2 + 2a^2$.
24. $9x^2 - 18ax - 7a^2$.
25. $10x^2 - 63x - 13$.
26. $x^4y^6 - 12x^2y^3 + 36$.
27. $4x^{2p} - 15x^p + 81$.
28. $2x^3 - 17bx^2 + 8b^2x$.
29. $4a^2 + 12ab + 9b^2$.
30. $5a^2x^2 - 2abx - 7b^2$.
31. $10x^4 - 15a^2x^3 - 100x^2a^4$.
32. $4a^{2m} + 16a^mb^n + 16b^{2n}$.
33. $4a^2x^2y^4 - 20abxy^2z + 25b^2z^2$.

32. Factoring the difference of squares. Under the method of the preceding paragraph we may factor the difference of squares.

Thus to factor $x^2 - b^2$ we observe that the product of the coefficient of x^2 and the constant term is

$$1 \cdot (-b^2) = -b^2.$$

Since the coefficient of x is zero, we have

$$-b + b = 0.$$

Hence $$(x + b)(x - b) = x^2 - b^2.$$

RULE. *Extract the square root of each term.*

The sum of these square roots is one factor, and their difference is the other.

EXAMPLE. Factor $9a^2x^6y^4 - 16b^8c^2$.

$$9a^2x^6y^4 - 16b^8c^2 = (3ax^3y^2 + 4b^4c)(3ax^3y^2 - 4b^4c).$$

33. Reduction to the difference of squares. The preceding method may be used when the expression to be factored becomes a perfect square by the addition of the square of some expression.

EXERCISES

Factor the following:

1. $a^4 + 4b^4$.

Solution:
$$\begin{aligned} a^4 + 4b^4 &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 \\ &= (a^2 + 2b^2)^2 - 4a^2b^2 \\ &= (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab). \end{aligned}$$

2. $1 - a^4$.
3. $a^4 + 4$.
4. $x^3 - x$.
5. $x^6y^4 + 4x^2$.
6. $4x^4 + y^4$.
7. $4a^2 - 25b^2$.
8. $x^4 + x^2 + 1$.
9. $16a^2b^4 - x^4$.
10. $x^4 + 9x^2 + 81$.
11. $4a^{2p} - 9b^2c^{2q}$.
12. $a^{2p+3} - 16a^3b^4$.
13. $x^{4n} + x^{2n} + 1$.
14. $36x^2y^4z^9 - 49u^2v^{16}$.
15. $x^4 - 13x^2 + 36$.
16. $my^4 + 16mx^4 - 12mx^2y^2$.
17. $9x^2 + 8x^2y^2 + 4y^4$.

34. Replacing a parenthesis by a letter. Any of the preceding methods may be applied when a polynomial appears in place of a letter in the expression to be factored. It is frequently desirable for simplicity to replace such a polynomial by a letter, and in the final result to restore the polynomial.

EXERCISES

Factor the following:

1. $2ax^2 - 2bx^2 - 6ax - 6bx + 8a - 8b$.

Solution:
$$\begin{aligned} &2ax^2 - 2bx^2 - 6ax - 6bx + 8a - 8b \\ &= 2(a-b)x^2 - 6(a-b)x - 8(a-b) \\ &= (a-b)(2x^2 - 6x - 8) \\ &= 2(a-b)(x^2 - 3x - 4) \\ &= 2(a-b)(x-4)(x+1). \end{aligned}$$

In this example the factor $(a - b)$ might have been replaced by a letter.

2. $a^2 + b^2 - c^2 - 9 - 2ab + 6c$.

Solution:
$$\begin{aligned} &a^2 + b^2 - c^2 - 9 - 2ab + 6c \\ &= a^2 - 2ab + b^2 - (c^2 - 6c + 9) \\ &= (a-b)^2 - (c-3)^2 \\ &= (a - b + c - 3)(a - b - c + 3). \end{aligned}$$

3. $(3x - y)(2a + p) - (3x - y)(a - q)$.
4. $(4a - 6b)(3m - 2p) + (a + 5b)(3m - 2p)$.
5. $(7a - 3y)(5c - 2d) - (6a - 2y)(5c - 2d)$.
6. $(x - y)(3a + 4b) - (4a - 5b)(x - y) - (x - y)(2a - 8b)$.

7. $6(x+y)^2 - 11(x+y) - 7$.
8. $4a^2 - 12ab + 9b^2 - x^2 - 2x - 1$.
9. $x^2a^2 + 2x^2a + x^2 - a^2 - 2a - 1$.
10. $ax^2 + 6ax + 9a - bx^2 - 6bx - 9b$.
11. $4(a-b)^2 - 5(a^2 - b^2) - 21(a+b)^2$.
12. $5(x+y)^2 - 12(x^2 - y^2) + 4(x-y)^2$.
13. $a^2b^2x^2 - a^2b^2 - 2abx^2 + 2ab + x^2 - 1$.
14. $(x-2y)(2a-3b) - (9b-10)(x-2y)$.

35. Factoring binomials of the form $a^n \pm b^n$. By § 27,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}).$$
$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1}),$$

where n is odd.

One can factor by inspection any binomial of the given form by reference to these equations.

EXERCISES

Factor the following:

1. $x^6 - y^6$.

Solution:
$$x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$$
$$= (x^2 + xy + y^2)(x - x)(x^2 - xy + y^2)(x + y).$$

2. $x^6 + 125$.
3. $x^7 - 1$.
4. $x^{12} - y^{12}$.
5. $x^9 - y^9$.
6. $x^{18} - y^{18}$.
7. $x^{16} - y^{16}$.
8. $a^2x^3 + a^5$.
9. $x^4 - a^8y^4$.
10. $216a + a^4$.
11. $ax^4 - 16a$.
12. $3a^7 - 96b^5a^2$.
13. $27x^6y^9 + 64y^3$.
14. $27x^5y^7 + x^2y^4$.
15. $16a^4b^8 - 81c^{16}d^8$.

36. Highest common factor. An expression that is not further divisible into factors with rational coefficients is called **prime.**

If two polynomials have the same expression as a factor, this expression is said to be their **common factor.**

The product of the common prime factors of two polynomials is called their **highest common factor,** or H.C.F.

The same common prime factor may occur more than once. Thus $(x-1)^2(x+1)$ and $(x-1)^2(x-2)^2$ have $(x-1)^2$ as their H.C.F.

37. H.C.F. of two polynomials. The process of finding the H.C.F. is performed as follows:

RULE. *Factor the polynomials. The product of the common prime factors is their H.C.F.*

EXERCISES

Find the H.C.F. of the following:

1. $4\,ab^2x^4 - 8\,ab^2x^2 + 4\,ab^2$ and $6\,abx^2 + 12\,abx + 6\,ab$.

Solution:

$$\begin{aligned} &4\,ab^2x^4 - 8\,ab^2x^2 + 4\,ab^2 \\ &= 4\,ab^2(x^4 - 2\,x^2 + 1) \\ &= 4\,ab^2(x-1)^2(x+1)^2. \end{aligned}$$

$$\begin{aligned} &6\,abx^2 + 12\,abx + 6\,ab \\ &= 6\,ab(x^2 + 2\,x + 1) \\ &= 6\,ab(x+1)^2. \end{aligned}$$

The H.C.F. is then $2\,ab(x+1)^2$.

2. $x^6 - y^6$ and $x^2 - y^2$.

3. $x^3 + x^2 - 12\,x$ and $x^2 + 5\,x + 4$.

4. $9\,mx^2 - 6\,mx + m$ and $9\,nx^2 - n$.

5. $6\,x - 4\,x^2 + 2\,ax - 3\,a$ and $9 - 4\,x^2$.

6. $12\,a^2 - 36\,ab + 27\,b^2$ and $8\,a^2 - 18\,b^2$.

7. $3\,a^2x - 6\,abx + 3\,b^2x$ and $4\,a^2y - 4\,b^2y$.

8. $2\,x - 4\,b - x^2 - 2\,bx$ and $4\,x - 5\,x^2 - 6$.

9. $6\,x^3 - 7\,ax^2 - 20\,a^2x$ and $3\,x^2 + ax - 4\,a^2$.

38. Euclid's method of finding the H.C.F. When one is unable to factor the polynomials whose H.C.F. is sought, the problem may nevertheless be solved by use of a method which in essence dates from **Euclid** (300 B.C.).

The validity of this process depends on the following

PRINCIPLE. *If a polynomial has a certain factor, any multiple of it has the same factor.*

Let $$x^n + Ax^{n-1} + Bx^{n-2} + \cdots + K$$
and $$x^m + ax^{m-1} + bx^{m-2} + \cdots + l$$

be represented by F and G respectively. The letters $A, B, \cdots, K$ and $a, b, \cdots, l$ represent integers, and m, the degree of G, is no greater than n, the degree of F. We seek a method of finding the H.C.F. of F and G if any exists. Call Q the quotient obtained by dividing F by G, and call R the remainder. Then (§ 26)

$$F = Q \cdot G + R, \tag{1}$$

where the degree of R in x is not so great as that of G. Now whatever the H.C.F. of F and G may be, it must also be the H.C.F. of G and R. For since

$$F - QG = R,$$

the H.C.F. of F and G must be a factor of the left-hand member, and hence a factor of R, which is equal to that member. Also every factor common to G and R must be contained in F, for any factor of G and R is a factor of the right-hand member of (1), and hence of F.

Thus our problem is reduced to finding the H.C.F. of G and R. Let Q_1 and R_1 be respectively the quotient and remainder obtained in dividing G by R.

Then $$G = Q_1R + R_1,$$

where the degree of R_1 in x is not as great as that of R. By reasoning similar to that just employed we see that the H.C.F. of G and R is also the H.C.F. of R and R_1. Continue this process of division.

Let $$R = Q_2R_1 + R_2,$$
$$R_1 = Q_3R_2 + R_3.$$
until, say in $$R_k = Q_{k+2}R_{k+1} + R_{k+3},$$

either R_k is exactly divisible by R_{k+1} (i.e. $R_{k+3} = 0$), or R_{k+3} does not contain x. This alternative must arise since the degrees in x of the successive remainders R, R_1, R_2, $\cdots$ are continually diminishing, and hence either the remainder must finally vanish or cease to contain x. Suppose $R_{k+3}=0$. Then the H.C.F. of R_k and R_{k+1} is R_{k+1} itself, which must, by the reasoning given above, be also the H.C.F. of F and G. If R_{k+3} does not contain x, then the H.C.F. of F and G, which must also be a factor of R_{k+3}, can contain no x, and must therefore be a constant.

Thus F and G have no common factor involving x.

This process is valid if the coefficients of F and G are rational expressions in any letters other than x.

39. Method of finding the H.C.F. of two polynomials. The above discussion we may express in the following

RULE. *Divide the polynomial of higher degree (if the degrees of the polynomials are unequal) by the other, and if there is a remainder, divide the divisor by it; if there is a remainder in this process, divide the previous remainder by it, and so on until either there is no remainder or it does not contain the letter of arrangement. If there is no remainder in the last division, the last divisor is the H.C.F. If the last remainder does not contain the letter of arrangement, then the polynomials have no common factor involving that letter.*

In the application of this rule any divisor or remainder may be multiplied or divided by any expression not involving the letter of arrangement without affecting the H.C.F.

EXERCISES

Find the H.C.F. of the following:

1. $2x^4 + 2x^3 - x^2 - 2x - 1$ and $x^4 + x^3 + 4x + 4$.

Solution:

$$
\begin{array}{lrl}
 & 2\,\underline{|}\,2x^4+2x^3-x^2-2x-1 & \underline{|x^4+x^3+4x+4} \\
 & \underline{2x^4+2x^3\qquad +8x+8} & \\
\text{Multiply by } -1, & -x^2-10x-9 & \\
 & x^2+10x+9\,|\,x^4+x^3+4x+4 & \underline{|x^2-9x+81} \\
 & \underline{x^4+10x^3+9x^2} & \\
 & -9x^3-9x^2+4x & \\
 & \underline{-9x^3-90x^2-81x} & \\
 & 81x^2+84x+4 & \\
 & \underline{81x^2+810x+729} & \\
\text{Divide by } -725, & -725x-725 &
\end{array}
$$

$$
\begin{array}{r}
x+9\,|\,x^2+10x+9\,\underline{|x+1} \\
\underline{x^2+x} \\
9x+9 \\
\underline{9x+9} \\
0\quad 0
\end{array}
$$

Thus the H.C.F. is $x + 1$.

This process may be performed in the following more compact form.

$$
\begin{array}{r|r|r|l}
2 & 2x^4+2x^3-x^2-2x-1 & x^4+x^3+4x+4 & x^2-9x+81 \\
 & 2x^4+2x^3\qquad +8x+8 & x^4+10x^3+9x^2 & \\
-1 & -x^2-10x-9 & -9x^3-9x^2+4x & \\
x+9 & x^2+10x+9 & -9x^3-90x^2-81x & \\
 & x^2+x & 81x^2+85x+4 & \\
 & 9x+9 & 81x^2+810x+729 & \\
 & 9x+9 & -725x-725 & -725 \\
 & 0\quad 0 & x+1 &
\end{array}
$$

Result: $x + 1$.

2. $x^2 + 6x - 7$ and $x^3 - 39x + 70$.

3. $x^5 - x^4 - x + 1$ and $5x^4 - 4x^3 - 1$.

4. $x^3 + 2x^2 + 9$ and $-5x^3 - 11x^2 + 15x + 9$.

5. $x^3 - 2x^2 - 15x + 36$ and $3x^2 - 4x - 15$.

6. $x^4 - 3x^3 + x^2 + 3x - 2$ and $4x^3 - 9x^2 + 2x + 3$.

7. $4x^3 - 18x^2 + 19x - 3$ and $2x^4 - 12x^3 + 19x^2 - 6x + 9$.

8. $x^4 + 4x^3 - 22x^2 - 4x + 21$ and $x^4 + 10x^3 + 20x^2 - 10x - 21$.

9. $6a^4x^3 - 9a^3x^2y - 10a^2xy^2 + 15ay^3$ and $10a^5x^6y^2 - 15a^4x^5y^3 + 8a^3x^4y^4 - 12a^2x^3y^5$.

40. Least common multiple. The **least common multiple** of two or more polynomials is the polynomial of least degree that contains them as factors. We may find the least common multiple of several polynomials by the following

RULE. *Multiply together all the factors of the various polynomials, each factor having the greatest exponent with which it appears in any of the polynomials.*

41. Second rule for finding the least common multiple. When only two polynomials are considered the previous rule is evidently equivalent to the following

RULE. *Multiply the polynomials together and divide the product by their highest common factor.*

EXERCISES

Find the least common multiple of the following:

1. $x^2 - y^2, x^2 + ay - ax - xy$, and $x^2 - 2xy + y^2$.

Solution:

$$x^2 - y^2 = (x - y)(x + y).$$
$$x^2 + ay - ax - xy = (x - y)(x - a).$$
$$x^2 - 2xy + y^2 = (x - y)^2.$$

Thus the L.C.M. $= (x - y)^2(x + y)(x - a)$.

2. $4a^2bc$, $6ab^2$, and $12c^2$.
3. $9xy^2$, $6x^2y^3$, and $3xy^2z^2$.
4. $(x + 1)(x^2 - 1)$ and $x^3 - 1$.
5. $x^4 + 4x^2y^2$ and $x^2 + 2y^2 - 2y$.
6. $4x^2 - 9y^2$ and $4x^2 - 12xy + 9y^2$.
7. $x^2 - 3x + 4$, $x^2 - 1$, and $x^2 - ax - x + a$.
8. $x - 1$, $2x^2 - 5x - 3$, and $2x^3 - 7x^2 + 2x + 3$.
9. $x^3 - 9x^2 + 26x - 24$ and $x^3 - 10x^2 + 31x - 30$.
10. $2x^2 - 3x - 9$, $x^2 - 6x + 9$, and $3x^2 - 9x - bx + 3b$.

CHAPTER III

FRACTIONS

42. General principles. The symbolic statements of the rules for the addition, subtraction, multiplication, and division of algebraic fractions are the same as the statements of the corresponding operations on numerical fractions given in (2), (3), and (4), § 6. This is immediately evident if we keep in mind the fact that algebraic expressions are symbols for numbers and that if the letters are replaced by numbers, the algebraic fraction becomes a numerical fraction.

43. PRINCIPLE I. *Both numerator and denominator of a fraction may be multiplied (or divided) by the same expression without changing the value of the fraction.*

This follows from (5), § 6.

44. PRINCIPLE II. *If the signs of both numerator and denominator of a fraction be changed, the sign of the fraction remains unchanged.*

This follows from Principle I, when we multiply both numerator and denominator by -1.

45. PRINCIPLE III. *If the sign of either numerator or denominator (but not both) be changed, the sign of the fraction is changed.*

This follows from (6), § 6.

46. Reduction. A fraction is said to be **reduced to its lowest terms** when its numerator and denominator have no common factor. We effect this reduction by the following

RULE. *Divide both numerator and denominator by their highest common factor.*

EXERCISES

Reduce the following to their lowest terms.

1. $\dfrac{12\,ax^2 - 12\,ab^2}{4\,ax^2 - 8\,abx + 4\,ab^2}$.

Solution: $$\frac{12\,ax^2 - 12\,ab^2}{4\,ax^2 - 8\,abx + 4\,ab^2} = \frac{12\,a(x-b)(x+b)}{4\,a(x-b)^2}.$$

H.C.F. $= 4\,a(x-b)$. $$= \frac{3(x+b)}{x-b}.$$

2. $\dfrac{a^3 - b^3}{a^2 - b^2}$.
3. $\dfrac{x^3 - 2\,x^2 + 2\,x}{x^5 + 4\,x}$.
4. $\dfrac{10\,x + 2\,ax - a - 5}{a - 2\,ax - 10\,x + 5}$.
5. $\dfrac{(a-b)^2}{a^2 - b^2}$.
6. $\dfrac{6\,x^2 - 8\,ax + 2\,a^2}{x^2 - a^2}$.
7. $\dfrac{2\,a^2b + 2\,ab^2 - 2\,abc}{3\,bc^2 - 3\,b^2c - 3\,abc}$.
8. $\dfrac{x^{12} - a^{12}}{x^{18} - a^{18}}$.
9. $\dfrac{a^2 + b^2 - c^2 + 2\,ab}{a^2 - b^2 + c^2 + 2\,ac}$.
10. $\dfrac{21\,x^3 - 9\,x^2 + 7\,x - 3}{3\,x^3 + 15\,x^2 + x + 5}$.
11. $\dfrac{2\,x^2 + 3\,x - 9}{x^2 - 9}$.
12. $\dfrac{x^4 - x^3 - x + 1}{2\,x^4 - x^3 - 2\,x + 1}$.
13. $\dfrac{x^3 + 3\,ax^2 + 3\,a^2x + a^3}{a^2 + 2\,ax + x^2}$.

47. Least common denominator of several fractions. We have the following

RULE. *Find the least common multiple of the various denominators.*

Multiply both numerator and denominator of each fraction by the expression which will make the new denominator the least common multiple of the denominators.

EXERCISES

Reduce the following to their least common denominator.

1. $\dfrac{2}{x}$, $\dfrac{3}{2\,x-1}$, and $\dfrac{2\,x-3}{4\,x^2-1}$.

Solution: The L.C.M. of the denominators is $x(4\,x^2-1)$. Thus the fractions are

$$\frac{2(4\,x^2-1)}{x(4\,x^2-1)},\ \frac{3\,x(2\,x+1)}{x(4\,x^2-1)},\ \text{and}\ \frac{x(2\,x-3)}{x(4\,x^2-1)}.$$

2. $\dfrac{a}{b-a}$, $\dfrac{c}{b+a}$, and $\dfrac{a^2+b^2}{a^2-b^2}$.
3. $\dfrac{x}{2\,x-3}$, $\dfrac{x+1}{4\,x^2+4\,x-15}$, and $\dfrac{x-2}{4\,x^2-25}$.

4. $\dfrac{1}{x^3 - y^3}$, $\dfrac{1}{x^4 - y^4}$, and $\dfrac{1}{x^2 - y^2}$.

5. $\dfrac{a}{a^2 - b^2}$, $\dfrac{b}{a^2 - ab}$, and $\dfrac{-a^2}{(a-b)^2}$.

6. $\dfrac{2x-1}{x^2 - 2x + 1}$, $\dfrac{x}{x^2 - 1}$, and $\dfrac{2x-3}{(x+1)^2}$.

7. $\dfrac{a}{a+b-c}$, $\dfrac{b}{a+b+c}$, and $\dfrac{c}{a^2 + 2ab + b^2 - c^2}$.

48. Addition of fractions. This operation we perform as follows:

RULE. *Reduce the fractions to be added to their least common denominator.*

Add the numerators for the numerator of the sum, and take the least common denominator for its denominator.

49. Subtraction of fractions. This operation we perform as follows:

RULE. *Reduce the fractions to their least common denominator.*

Subtract the numerator of the subtrahend from that of the minuend for the numerator of the result, and take the least common denominator for its denominator.

50. Multiplication of fractions. This operation we perform as follows:

RULE. *Multiply the numerators together for the numerator of the product, and the denominators for its denominator.*

51. Division of fractions. This operation we perform as follows:

RULE. *Invert the terms of the divisor and multiply by the dividend.*

REMARK. Since a fraction is a means of indicating division, $\frac{a}{b} \div \frac{c}{d}$ and $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are two expressions for the same thing.

EXERCISES

Perform the indicated operations and bring the results into their simplest forms.

1. $\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$

Solution: $$\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}=\dfrac{\dfrac{(a+b)^2+(a-b)^2}{a^2-b^2}}{\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}}$$
$$=\frac{(a+b)^2+(a-b)^2}{a^2-b^2}\cdot\frac{a^2-b^2}{(a+b)^2-(a-b)^2}$$
$$=\frac{2a^2+2b^2}{4ab}=\frac{a^2+b^2}{2ab}.$$

2. $\dfrac{\frac{7}{3}-1}{\frac{4}{3}+1}.$

3. $\dfrac{(\frac{24}{7})^2-1}{2\cdot\frac{24}{7}}.$

4. $\dfrac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}.$

5. $\dfrac{2\cdot\frac{12}{5}}{1-(\frac{12}{5})^2}.$

6. $\dfrac{3+\frac{5}{3}+\frac{7}{9}}{\frac{4}{3}+\frac{2}{9}+\frac{1}{27}}.$

7. $\dfrac{2+\frac{4}{3}+\frac{8}{9}}{1-\frac{2}{3}+\frac{4}{9}}.$

8. $\dfrac{a}{b}-\dfrac{a+c}{2b}.$

9. $\dfrac{x}{ab}-\dfrac{y}{ac}-\dfrac{z}{bc}.$

10. $\dfrac{1}{a-b}+\dfrac{1}{a+b}.$

11. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}.$

12. $\dfrac{a-2b}{c}+\dfrac{3b}{c}.$

13. $\dfrac{2x}{a-1}-\dfrac{x}{a^2-1}.$

14. $\dfrac{3x-1}{1-3x}-\dfrac{2x-7}{7}.$

15. $\dfrac{2x-1}{x-2}-\dfrac{2x-5}{x-4}.$

16. $\dfrac{5}{4x-4}-\dfrac{7}{6x+6}.$

17. $\dfrac{5}{3x-9}-\dfrac{8}{5x-15}.$

18. $\dfrac{a+b}{a-b}\div\dfrac{a^2+2ab-b^2}{a^2-b^2}.$

19. $\dfrac{2a-3b}{12a}+\dfrac{3a-2b}{15a}.$

20. $\dfrac{8}{15(x-1)}+\dfrac{9}{10(x+1)}.$

21. $\dfrac{ad+bc}{2cd(c-d)}+\dfrac{ad-bc}{2cd(c+d)}.$

22. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4xy}{x^2-y^2}.$

23. $\dfrac{x^2+x(a+b)+ab}{x^2-x(a+b)+ab}\cdot\dfrac{x^2-a^2}{x^2-b^2}.$

24. $\dfrac{5z}{6a}+\dfrac{3z}{14a}+\dfrac{18z}{35a}-\dfrac{2z}{15a}.$

25. $\dfrac{5}{a^2-9a+14}\div\dfrac{3}{a^2-5a-14}.$

26. $\left(\dfrac{x^2}{y^3}+\dfrac{1}{x}\right)\div\left(\dfrac{x}{y^2}-\dfrac{1}{y}+\dfrac{1}{x}\right).$

27. $\dfrac{a(a-x)}{a^2+2ax+x^2}\cdot\dfrac{a(a+x)}{a^2-2ax+x^2}.$

28. $\dfrac{1-a^2}{1+b}\cdot\dfrac{1-b^2}{a+a^2}\cdot\left(1+\dfrac{a}{1-a}\right).$

29. $\dfrac{2}{(x-1)^3}+\dfrac{1}{(x-1)^2}+\dfrac{2}{x-1}-\dfrac{1}{x}.$

30. $a+\dfrac{b}{c+\dfrac{d}{e}}$.

31. $\dfrac{\left(\dfrac{a}{b}\right)^3-1}{\left(\dfrac{a}{b}\right)^2-1}$.

32. $\dfrac{x^2-\dfrac{1}{x}}{x+\dfrac{1}{x}+1}$.

33. $\dfrac{1}{x+\dfrac{1}{1-\dfrac{x+1}{x-3}}}$.

34. $\dfrac{3}{4+\dfrac{1}{5+\frac{1}{2}}}$.

35. $a+\dfrac{b}{c+\dfrac{d}{e+\dfrac{f}{g}}}$.

36. $\dfrac{\dfrac{a}{a+b}+\dfrac{b}{a-b}+\dfrac{ab}{a^2-b^2}}{\dfrac{1}{(a+b)^2}+\dfrac{1}{(a-b)^2}}$.

37. $\dfrac{\dfrac{1}{1+x}+\dfrac{1}{1-x}}{\dfrac{1}{1-x}-\dfrac{1}{1+x}}$.

38. $\dfrac{\dfrac{a^2+b^2}{a}-b}{\dfrac{1}{a}-\dfrac{1}{b}}\cdot\dfrac{a^2-b^2}{a^3+b^3}$.

39. $\dfrac{x-3y}{x^2-2xy-15y^2}\div\dfrac{x+3y}{x^2-8xy+15y^2}$.

40. $\dfrac{1}{3(x+1)}-\dfrac{x+2}{3(-4-3x+x^2)}$.

41. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y+z}}\left(1+\dfrac{y^2+z^2-x^2}{2yz}\right)$.

42. $\dfrac{2a-3b+4}{6}-\dfrac{3a-4b+5}{8}+\dfrac{a-1}{12}$.

43. $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

44. $\dfrac{3xyz}{yz+zx-xy}-\dfrac{\dfrac{x-1}{x}+\dfrac{y-1}{y}+\dfrac{z-1}{z}}{\dfrac{1}{x}+\dfrac{1}{y}-\dfrac{1}{z}}$.

45. $\left(\dfrac{2x+y}{x+y}+\dfrac{2y-x}{x-y}-\dfrac{x^2}{x^2-y^2}\right)\div\dfrac{x^2+y^2}{x^2-y^2}$.

46. $\dfrac{a-3b}{6a}+\dfrac{4a-b}{2b}+\dfrac{5a+3c}{9c}-\dfrac{a^2-bc}{2ac}-\dfrac{2a}{b}$.

47. $\dfrac{bcd}{(a-b)(a-c)(a-d)}+\dfrac{cda}{(b-c)(b-d)(b-a)}$
$+\dfrac{dab}{(c-b)(c-a)(c-d)}$
$+\dfrac{abc}{(d-a)(d-b)(d-c)}$.

48. $\dfrac{1}{6a}+\dfrac{a-2b}{3ab}-\dfrac{3}{4b}-\dfrac{3a-4c^2}{8ac^2}+\dfrac{9}{8c^2}+\dfrac{5c^2-5b}{12bc^2}$.

49. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{1}{(x-1)^2}+\dfrac{2}{(x+1)^2}-\dfrac{4}{x^2-1}-\dfrac{4}{(x^2-1)^2}$.

CHAPTER IV

EQUATIONS

52. Introduction. An **equation** is a statement of equality between two expressions.

We assume the following

AXIOM. *If equals be added to, subtracted from, multiplied by, or divided by equals, the results are equal.*

As always, we exclude division by zero. In dividing an equation by an algebraic expression one must always note for what values of the letters the divisor vanishes and exclude those values from the discussion.

53. Identities and equations of condition. Equations are of two kinds:

First. Equations that may be reduced to the equation $1 = 1$ by performing the indicated operations are called **identities.**

Thus
$$2 = 2,$$
$$a - b = (3a - 2b) - (2a - b)$$

are equations of this type. In identities the sign $=$ is often replaced by $\equiv$. It should be noted that *identities are true whatever numerical values the letters may have.*

Second. Equations that cannot be reduced to the form $1 = 1$, but which are true only when some of the letters have particular values, are called **equations of condition** or simply **equations.**

Thus $x = 2$ cannot further be simplified, and is true only when x has the value 2. Also $x = 2a$ is true only when x has the value $2a$ or a has the value $\frac{x}{2}$. If in this equation x is replaced by $2a$, the equation of condition reduces to an identity.

The number or expression which on being substituted for a letter in an equation reduces it to an identity is said to **satisfy** the equation.

Thus the number 5 satisfies the equation $x^2 - 24 = 1$. The number 3 satisfies the equation $(x - 3)(x + 4) = 0$.

The process of finding values that satisfy an equation is called **solving** the equation. The development of methods for the solutions of the various forms of equations is the most important question that algebra considers.

In an equation in which there are two letters it may be possible to find a value which substituted for either will satisfy the equation. Thus the equation $x - 2a = 0$ is satisfied if x is replaced by $2a$, or if a is replaced by $\frac{x}{2}$. In the former case we have solved for x, that is, have found a value that substituted for x satisfies the equation. In the latter case we have solved for a. In any equation it is necessary to know which letter we seek to replace by a value that will satisfy the equation, that is, *with respect to which letter* we shall solve the equation.

The letter with respect to which we solve an equation is called the **variable.**

Values which substituted for the variable satisfy the equation are called **roots** or **solutions** of the equation.

When only one letter, i.e. the variable, occurs in an equation, the root is a number. When letters other than the variable occur, the root is expressed in terms of those letters.

54. Linear equations in one variable. An equation in which the variable occurs only to the first degree is called a **linear equation.** To solve a linear equation in one variable we apply the following

RULE. *Apply the axiom* (§ 52) *to obtain an equation in which the variable is alone on the left-hand side of the equation.*

The right-hand side is the desired solution.

To test the accuracy of the work substitute the solution in the original equation and reduce to the identity $1 = 1$.

Since the result of adding two numbers is a definite number, and the same is true for the other operations used in finding the solution of a linear equation, it appears that every linear equation in one variable has one and only one root.

When both sides of an equation have a common denominator, the numerators are equal to each other. This appears from multiplying both sides of the equation by the common denominator and then canceling it from both fractions.

EXERCISES

Solve:

1. $\dfrac{4x-2}{5}+\dfrac{5x}{8}=\dfrac{3x}{4}+5.$

Solution: Transpose the term involving x,

$$\frac{4x-2}{5}+\frac{5x}{8}-\frac{3x}{4}=5.$$

Add fractions, $$\frac{32x-16+25x-30x}{40}=5.$$

Clear of fractions and simplify, $$27x=216.$$
$$x=8.$$

2. $\dfrac{a(d^2+x^2)}{dx}=ac+\dfrac{ax}{d}.$

Solution: Divide by a, $$\frac{d^2+x^2}{dx}=c+\frac{x}{d}.$$

Transpose the term involving x,

$$\frac{d^2+x^2}{dx}-\frac{x}{d}=c.$$

Add fractions, $$\frac{d^2+x^2-x^2}{dx}=c.$$

Clear of fractions and simplify, $$d=cx.$$
$$x=\frac{d}{c}.$$

3. $(a-1)x=b-x.$
4. $(a-x)(1-x)=x^2-1.$
5. $a(x-a^2)=b(x-b^2).$
6. $2x-\frac{3}{5}x=\frac{3}{2}x-\frac{1}{2}-\frac{2}{5}x+2.$
7. $8x-7+x=9x-3-4x.$
8. $.617x-.617=12.34-1.234x.$
9. $3(2x-.3)=.6+5(x-.1).$
10. $7-5x+10+8x-7+3x=x.$
11. $(x-3)(x-4)=(x-6)(x-2).$
12. $\frac{4}{7}\{\frac{5}{12}[\frac{7}{8}(\frac{3}{4}x+5)-10]+3\}-8=0.$
13. $(1+6x)^2+(2+8x)^2=(1+10x)^2.$
14. $5=3x+\frac{2}{3}(x+3)-\frac{1}{2}(11x-37).$
15. $2(x+5)(x+2)=(2x+7)(x+3).$
16. $(7\frac{1}{3}x-2\frac{1}{2})-[4\frac{1}{2}-\frac{1}{2}(3\frac{1}{3}-5x)]=18\frac{1}{3}.$
17. $6x-7(11-x)+11=4x-3(20-x).$
18. $(a-b)(x-c)+(a+b)(x+c)=2(bx+ad).$
19. $2x-3(5+\frac{3}{4}x)+\frac{2}{3}(4-x)-\frac{1}{4}(3x-16)=0.$
20. $5x-2=\frac{2}{3}x+\frac{3}{4}x+\frac{4}{5}x+\frac{9}{10}x+\frac{11}{12}x+\frac{14}{15}x.$
21. $(a-b)(a-c+x)+(a+b)(a+c-x)=2a^2.$
22. $12.9x-1.45x-3.29-.99x-11x+.32=0.$
23. $5.7x-2\frac{1}{3}(7.8-9.3x)=5.38-4\frac{3}{4}(.28+3.6x).$

24. $3 - \frac{1}{3} = \frac{1}{\frac{1}{3} + \frac{1}{x}}$.

25. $\frac{a}{mx} + \frac{b}{nx} = c$.

26. $\frac{x-1}{x-3} = \frac{x-4}{x-5}$.

27. $\frac{\frac{4}{5}(4x-1)}{\frac{3}{4}(5x+1)} = \frac{2}{3}$.

28. $\frac{\frac{2}{5}(x-4)}{\frac{3}{8}(3x+5)} = \frac{1}{6}$.

29. $\frac{2x^2 - 3x + 5}{7x^2 - 4x - 2} = \frac{2}{7}$.

30. $\frac{9}{x} + \frac{1}{2} = \frac{10}{x} + \frac{4}{9}$.

31. $\frac{\frac{1}{4} - x}{\frac{1}{4} + x} + \frac{1}{4} = \frac{x}{\frac{1}{4} + x} - \frac{1}{4}$.

32. $\frac{a + bx}{a + b} = \frac{c + dx}{c + d}$.

33. $\frac{2}{3} \cdot \frac{5x - 2}{7x - 3} = \frac{5}{7} \cdot \frac{2x - 5}{3x - 7}$.

34. $\frac{ax}{b} + \frac{cx}{d} + \frac{fx}{g} = h$.

35. $\frac{x + a}{b} - \frac{b}{a} = \frac{x - b}{a} + \frac{a}{b}$.

36. $\frac{x - a}{a} + b = x - 1$.

37. $\frac{3x - 19}{x - 13} + \frac{5x - 25}{x + 7} = 8$.

38. $\frac{\frac{3}{2} - \frac{1}{x}}{\frac{3}{2} + \frac{1}{x}} - \frac{\frac{2}{3} - \frac{1}{x}}{\frac{2}{3} + \frac{1}{x}} = \frac{\frac{3}{2} - \frac{2}{3}}{\frac{2}{3} \cdot \frac{1}{x} + 1}$.

39. $\frac{a - x}{a} + \frac{b - x}{b} + \frac{c - x}{c} = 0$.

40. $\frac{x - 8}{x + 2} + \frac{x + 12}{x - 8} = 2 + \frac{18}{x + 2}$.

41. $\frac{a(2x + 1)}{3b} - \frac{5ax - 4b}{5b} = \frac{4}{5}$.

42. $\frac{ax}{b} + \frac{bx}{a} + \frac{2ab}{a + b} = \frac{(a + b)^2 x}{ab}$.

43. $8\frac{1}{4}x - \frac{x}{5} - 3\frac{2}{3}x - 4\frac{1}{5}x + 1 = 0$.

44. $\frac{1}{a + b} + \frac{a + b}{x} = \frac{1}{a - b} + \frac{a - b}{x}$.

45. $\frac{2(x - 1)}{x - 7} + \frac{x + 8}{x - 4} = \frac{3(5x + 16)}{5x - 28}$.

46. $\frac{a^2b - x}{a} + \frac{b^2c - x}{b} + \frac{ac^2 - x}{c} = 0$.

47. $\frac{5x - .4}{.3} + \frac{1.3 - 3x}{2} = \frac{1.8 - 8x}{1.2}$.

48. $\frac{8x}{6x + 2} + \frac{7x}{15x + 5} + \frac{x}{3x + 1} = 2$.

49. $\frac{7x - 2}{3} - \frac{4}{5}(x + 3) + 6 = \frac{3(x + 2)}{2}$.

50. $\dfrac{5x-1}{3(x+1)}-\dfrac{3x+2}{2(x-1)}=\dfrac{x^2-30x+2}{6x^2-6}.$

51. $\dfrac{3x-2}{x+3}+\dfrac{7x-3}{x+2}+\dfrac{x+100}{x^2+5x+6}=10.$

52. $\dfrac{3b(x-a)}{5a}+\dfrac{x-b^2}{15b}+\dfrac{b(4a+cx)}{6a}=0.$

53. $\dfrac{16x-27}{21}-\dfrac{x+3}{5}=\dfrac{5+3x}{2}-\dfrac{4x-7}{3}.$

54. $\dfrac{a(b-x)}{bx}+\dfrac{b(c-x)}{cx}=\dfrac{a+b}{x}+\left(\dfrac{b}{c}+\dfrac{a}{b}\right).$

55. $\dfrac{5x-6}{10}-\dfrac{9-10x}{14}=\dfrac{3x-4}{5}-\dfrac{3-4x}{7}.$

56. $\dfrac{4-2x}{3}-\dfrac{4}{6x-3}=\dfrac{1.5x}{x-.5}-\dfrac{4x^2}{3(2x-1)}.$

57. $3-\dfrac{1}{6(2x-5)}=\dfrac{1}{2(2x-5)}+\dfrac{7}{3(2x-5)}.$

58. $\dfrac{x^{n+1}-3x^{n-1}}{4x}-\dfrac{3x^{n-1}-x^n}{4}=\dfrac{x^n}{2}-3x^{n-2}.$

59. $\dfrac{ax-bc}{ab}-\dfrac{bx-ac}{c^2}=\dfrac{cx-b^2}{bc}-\dfrac{x-a}{c}+1-\dfrac{x}{a}.$

60. $\dfrac{x-2a}{b+c-a}+\dfrac{x-2b}{a+c-b}+\dfrac{x-2c}{a+b-c}=\dfrac{3x}{a+b+c}.$

61. $\dfrac{2x^n+7x^{n-1}}{9}+\dfrac{7x^n-44x^{n-1}}{5x-14}=\dfrac{4x^n+27x^{n-1}}{18}.$

62. $\dfrac{3x+3}{2}-\left(\dfrac{x+1}{6}+3\right)=\dfrac{5x+2}{3}-\left(\dfrac{3x-1}{2}-3\right).$

63. $\dfrac{a(x-3)}{b}+\dfrac{b(x-3)}{a}+\dfrac{a^2(x-1)}{b^2}+\dfrac{b^2(x-1)}{a^2}=4.$

64. $\dfrac{ac}{m(a-b)b}-\dfrac{(m+n)^2x}{mb}-\dfrac{nx}{b}=\dfrac{c}{m(a-b)}-\dfrac{3nx}{b}.$

65. $\dfrac{4(13x-.6)}{5}+\dfrac{3(1.2-x)}{2}=\dfrac{9x+.2}{20}+\dfrac{5+7x}{4}+x.$

66. $\dfrac{(a^2+b^2)}{b}(x-a)+\dfrac{a^2-b^2}{a}(x-b)=2a(2a+b-x).$

67. $\dfrac{ax-b}{mx-p}+\dfrac{cx-d}{nx-q}+\dfrac{(bn+dm)x+(bp+dq)}{(mx-p)(nx-q)}=\dfrac{a}{m}+\dfrac{c}{n}.$

55. Solution of problems. The essential step in solving a **problem by algebra** is the expression of the conditions of the problem by algebraic symbols. This is, in fact, nothing else than a translation of the problem from the English language into the language of algebra. The translation should be made as close as possible, clause by clause in most cases. In general the result sought should be represented by the variable, which for that reason is often called the **unknown quantity.**

EXAMPLE. What number is it whose third part exceeds its fourth part by sixteen?

Solution: "What number is it" is translated by x. Thus we let x represent the number sought. "Whose third part" is translated by $\frac{x}{3}$. "Exceeds its fourth part" is translated by $\frac{x}{3} - \frac{x}{4}$, i.e. the third part less the fourth part leaves something. "By sixteen" gives us the amount of the remainder. Thus the translation of the problem into algebraic language is

Let x represent the number sought.

$$\frac{x}{3} - \frac{x}{4} = 16.$$

This equation should be solved and checked by the methods already given.

PROBLEMS

1. What number is it whose third and fifth parts together make 88?

2. What number increased by 3 times itself and 5 times itself gives 99?

3. What is the number whose third, fourth, sixth, and eighth parts together are 3 less than the number itself?

4. What number is it whose double is 7 more than its fourth part?

5. In 10 years a young man will be $3\frac{1}{3}$ times as old as his brother is now. The brother is $7\frac{1}{2}$ years old. How old is the young man?

6. A father who is 53 years old is 3 years more than $12\frac{1}{2}$ times as old as his son. How old is the son?

7. If you can tell how many apples I have in my basket, you may have 4 more than $\frac{1}{7}$, or, what is the same thing, 4 less than $\frac{1}{3}$ of them. How many have I?

8. If Mr. A received $\frac{1}{6}$ more salary than at present, he would receive \$2100. How much does he receive?

9. A boy spends $\frac{1}{3}$ of his money in one store and $\frac{1}{7}$ of what remains in another, and has 24 cents left. How much had he?

10. A man who is 3 months past his fifty-fifth birthday is $4\frac{1}{4}$ times as old as his son. How old is the son?

11. In a school are four classes. In the first is $\frac{1}{5}$ of all the pupils; in the second, $\frac{1}{4}$; in the third, $\frac{2}{7}$; in the fourth, 37. How many pupils are in the school?

12. A merchant sold to successive customers $\frac{1}{9}$, $\frac{1}{6}$, and $\frac{1}{4}$ of the original length of a piece of cloth. He had left 2 yards less than half. How long was the piece?

13. How may one divide 77 into two parts of which one is $2\frac{1}{2}$ times as great as the other?

14. The sum of two numbers is 73 and their difference is 15. What are the numbers?

15. A father is $4\frac{1}{2}$ times as old as his son. Father and son together are 27 years younger than the grandfather, who is 71 years old. How old are father and son?

16. The sum of two numbers is 999. If one divides the first by 9 and the second by 6, the sum of these quotients is 138. What are the numbers?

17. The first of two numbers whose sum is a is b times the second. What are the numbers?

18. If the city of A had 14,400 more inhabitants, it would have 3 times as many as the city of B. Both A and B have together 12,800 more than the city of C, where there are 172,800 inhabitants. How many are in A and B?

19. Two men who are 25 miles apart walk toward each other at the rates of $3\frac{1}{2}$ and 4 miles an hour respectively. After how long do they meet?

20. A courier leaves a town riding at the rate of 6 miles an hour. Seven hours later a second courier follows him at the rate of 10 miles an hour. How soon is the first overtaken?

21. A can copy 14 sheets of manuscript a day. When he had been working 6 days, B began, copying 18 sheets daily. How many sheets had each written when B had finished as many as A?

22. The pendulum of a clock swings 387 times in 5 minutes, while that of a second clock swings 341 times in 3 minutes. After how long will the second have swung 1632 times more than the first?

23. The difference in the squares of two numbers is 221. Their sum is 17. What are the numbers?

24. If a book had 236 more pages it would have as many over 400 pages as it now lacks of that number. How many pages has the book?

25. A man is now 63 years old and his son 21. When was the father 19 times as old as his son?

26. If 7 oranges cost as much less than 50 cents as 13 do more than 50 cents, how much do they cost apiece?

27. The numerator of a fraction is 6 less than the denominator. Diminish both numerator and denominator by 1 and the fraction equals $\frac{1}{2}$. Find the fraction.

28. The sum of three numbers is 100. The first and second are respectively 9 and 7 greater than the third. What are the numbers?

29. Out of 19 people there were $\frac{2}{3}$ as many children as women, and $1\frac{1}{2}$ times as many men as women. How many were there of each?

30. A boy has twice as many brothers as sisters. His sister has 5 times as many brothers as sisters. How many sons and daughters were there?

31. A dealer has 5000 gallons of alcohol which is 85% pure. He wishes to add water so that it will be 75% pure. How much water must he add?

32. How much water must be added to 5 quarts of acid which is 10% full strength to make the mixture $8\frac{1}{3}$% full strength?

33. A merchant estimated that his supply of coffee would last 12 weeks. He sold on the average 18 pounds a week more than he expected, and it lasted him 10 weeks. How much did he have?

34. At what time between 3 and 4 o'clock are the hands of a clock pointing in the same direction?

35. At what time between 11 and 12 o'clock is the minute hand at right angles to the hour hand?

36. A merchant bought cloth for $2 a yard, which he was obliged to sell for $1.75 a yard. Since the piece contained 3 yards more than he expected, he lost only 2%. How many yards actually in the piece?

37. A man has three casks. If he fills the second out of the first, the latter is still $\frac{3}{5}$ full. If he fills the third out of the second, the latter is still $\frac{1}{6}$ full. The second and third together hold 100 quarts less than the first. How much does each hold?

38. A crew that can cover 4 miles in 20 minutes if the water is still, can row a mile downstream in $\frac{2}{3}$ the time that it can row the mile upstream. How rapid is the stream?

39. A cask is emptied by three taps, the first of which could empty it in 20 minutes, the second in 30 minutes, the third in 35 minutes. How long is required for all three to empty the cask?

40. A can dig a trench in $\frac{2}{3}$ the time that B can; B can dig it in $\frac{4}{5}$ the time that C can; and A and C can dig it in 8 days. How long is required by all working together?

56. Linear equations in two variables. A simple equation in one variable has one and only one solution, as we have already seen (p. 33). On the other hand, an equation of the first degree in two variables has many solutions.

For example,
$$3x + 7y = 1$$
is satisfied by innumerable pairs of numbers which may be substituted for x and y. For, transposing the term in y, we get
$$x = \frac{1 - 7y}{3},$$
from which it appears that when y has any particular numerical value the equation becomes a linear equation in x alone, and hence has a solution. Thus, when $y = 1$, $x = -2$, and this pair of values is a solution of the equation. Similarly, $x = -9$, $y = 4$ also satisfy the equation.

57. Solution of a pair of equations. If in solving the equation just considered, the values of x and y that one may use are no longer unrestricted in range, but must also satisfy a second linear equation, we get usually only a single pair of solutions. Thus if we seek a solution, that is, a pair of values of x and y satisfying
$$3x + 7y = 1,$$
such that also
$$x + y = -1,$$
we find that the pair of values $x = -2$, $y = 1$ satisfy both equations. Any other solution of the first equation, as, for instance, $x = 9$, $y = -4$, does not obey the condition imposed by the second.

Two equations which are not reducible to the same form are called **independent.**

Thus
$$6x - 8y - 4 = 0$$
and
$$3x - 4y = 2$$
are not independent, since the first is readily reduced to the second by transposing and dividing by 2. They are, in fact, essentially the same equation. On the other hand,
$$x - 4y = 2$$
and
$$3x - 4y = 2$$
are not reducible to the same form and are independent. Since dependent equations are identical except for the arrangement of terms and some constant factor, all their solutions are common to each other.

This principle we may state as follows:

Two equations

$$ax + by + c = 0$$

and

$$a'x + b'y + c' = 0$$

are dependent when and only when

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Independent equations in more than one variable which have a common solution are called **simultaneous** equations.

Two pairs of simultaneous equations which are satisfied by the same pair (or pairs) of values of x and y and only these are called **equivalent**.

Thus $\begin{cases} 3x + 7x = 1, \\ x + y = -1 \end{cases}$ **and** $\begin{cases} x = -2, \\ y = 1 \end{cases}$

are equivalent pairs of equations.

58. Independent equations. We now prove the following

THEOREM. *If $A = 0$ and $B = 0$ represent two independent equations, then the pairs of equations*

$$\begin{cases} A = 0, \\ B = 0 \end{cases} (1) \quad \textit{and} \quad \begin{cases} aA + bB = 0, \\ cA + dB = 0 \end{cases} \qquad (2)$$

are equivalent where a, b, c, and d are any numbers such that $ad - bc$ is not equal to zero.

The letters A and B symbolize linear expressions in x and y. Evidently any pair of values of x and y that makes both $A \equiv 0$ and $B \equiv 0$, i.e. satisfies (1), also makes $aA + bB \equiv 0$ and $cA + dB \equiv 0$, i.e. also satisfies (2). We must also show that any values of x and y that satisfy (2) also satisfy (1).

For a certain pair of values of x and y let

$$aA + bB \equiv 0, \qquad (3)$$

$$cA + dB \equiv 0. \qquad (4)$$

Multiply (3) by c and (4) by a (§ 52).

Then

$$acA + bcB \equiv 0, \qquad (5)$$

$$acA + adB \equiv 0. \qquad (6)$$

Subtract (5) from (6) (§ 52),

$$(ad - bc)B \equiv 0.$$

Thus, by § 5, either $ad - bc \equiv 0$ or $B \equiv 0$.

But $ad - bc$ is not zero, by hypothesis; consequently $B \equiv 0$. Similarly we could show that $A \equiv 0$.

Thus if we seek the solution of a pair of equations $A = 0$, $B = 0$, we may obtain by use of this theorem a pair of equivalent equations whose solution is evident, and find immediately the solution of the original equations.

59. Solution of a pair of simultaneous linear equations. The foregoing theorem affords the following

RULE. *Multiply each of the equations by some number such that the coefficients of one of the variables in the resulting pair of equations are identical.*

Subtract one equation from the other and solve the resulting simple equation in one variable.

Find the value of the other variable by substituting the value just found in one of the original equations.

Check the result by substituting the values found for both variables in the other equation.

EXAMPLE.

Solve
$$3x + 7y = 1, \quad (1)$$
$$x + y = -1. \quad (2)$$

Solution: Multiply (1) by 1 and (2) by 3,

$$3x + 7y = 1,$$
$$3x + 3y = -3$$

Subtract,
$$4y = 4$$
$$y = 1.$$

Substitute in (2),
$$x + 1 = -1.$$
$$x = -2.$$

Check: Substitute in (1),

$$3 \cdot (-2) + 7 \cdot 1 = -6 + 7 = 1.$$

60. Incompatible equations. Equations in more than one variable that do not have any common solution are called **incompatible.**

THEOREM. *The equations*

$$ax + by = c, \tag{1}$$

$$a'x + b'y = c' \tag{2}$$

are incompatible when and only when $ab' - ba' = 0$.

Apply the rule of § 59 to find the solution of these equations. Multiply (1) by a' and (2) by a.

We obtain $\qquad aa'x + a'by = ca',$

$$aa'x + ab'y = ac'.$$

Subtract, $\qquad (ab' - a'b)\,y = ac' - ca'.$

If now $ab' - a'b$ is not zero, we get a value of y; but since under our hypothesis $ab' - a'b = 0$, we can get no value for y since division by zero is ruled out (§ 7). Thus no solution of (1) and (2) exists.

EXAMPLE.

Solve
$$3x + 7y = 1, \tag{1}$$
$$6x + 14y = 1. \tag{2}$$

Solution: Multiply (1) by 2,

$$6x + 14y = 2$$
$$6x + 14y = 1$$

Subtract, $\qquad 0 = 1$

which is absurd. Thus no solution exists.

61. Résumé. We observe that pairs of equations of the form

$$ax + by + c = 0,$$
$$a'x + b'y + c' = 0$$

fall into three classes:

(a) *Dependent equations,* which have innumerable common solutions.

Then $$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}. \tag{1}$$

(b) *Incompatible equations,* which have no common solution.

Then $\qquad ab' - a'b = 0$, but (1) is not true.

(c) *Simultaneous equations,* which have one and only one pair of solutions.

Then $\qquad ab' - a'b \neq 0.$

EXERCISES

Solve and check the following:

1. $2x + 5y = 1,$ $6x + 7y = 3.$

2. $4x - 6y = 8,$ $\frac{2}{3}x - y = \frac{4}{3}.$

3. $6x + 8y = 18,$ $x + \frac{4}{3}y = 3.$

4. $7x - 3y = 27,$ $5x - 6y = 0.$

5. $2x - \frac{7}{3}y = 4,$ $3x - \frac{7}{2}y = 0.$

6. $\frac{1}{3}y = \frac{1}{2}x - 1,$ $\frac{4}{15}y = \frac{2}{5}x - 1.$

7. $5x - 4y + 1 = 0,$ $1.7x - 2.2y + 7.9 = 0.$

8. $3x + 4y = 253,$ $y = 5x.$

9. $5x + 3y + 2 = 0,$ $3x + 2y + 1 = 0.$

10. $x + my = a,$ $x - ny = b.$

11. $x + y = \frac{1}{2}(5a + b),$ $x - y = \frac{1}{2}(a + 5b).$

12. $2x - 3y = -5a,$ $3x - 2y = -5b.$

13. $\frac{3}{4}x - \frac{1}{2}(y + 1) = 1,$ $\frac{1}{3}(x + 1) + \frac{3}{4}(y - 1) = 9.$

14. $3.5x + 2\frac{1}{3}y = 13 + 4\frac{1}{7}x - 3.5y,$ $2\frac{1}{7}x + .8y = 22\frac{1}{2} + .7x - 3\frac{1}{3}y.$

15. $3x + 2y = 5a^2 + ab + 5b^2,$ $3y + 2x = 5a^2 - ab + 5b^2.$

16. $\dfrac{3}{x} + \dfrac{8}{y} = 3,$ $\dfrac{15}{x} - \dfrac{4}{y} = 4.$

17. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6},$ $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{6}.$

HINT. Retain fractions.

18. $\dfrac{x - c}{y - c} = \dfrac{a}{b},$ $x - y = a - b.$

19. $\dfrac{3x + 1}{4 - 2y} = \dfrac{4}{3},$ $x + y = 1.$

20. $\dfrac{x + 2y + 1}{2x - y + 1} = 2,$ $\dfrac{3x - y + 1}{x - y + 3} = 5.$

21. $\dfrac{x}{a} + \dfrac{y}{b} = c,$ $\dfrac{x}{a_1} + \dfrac{x}{b_1} = c_1.$

22. $\dfrac{5}{x + 2y} = \dfrac{7}{2x + y},$ $\dfrac{7}{3x - 2} = \dfrac{5}{6 - y}.$

23. $\dfrac{x + 1}{y + 1} = \dfrac{a + b + c}{a - b + c},$ $\dfrac{x - 1}{y - 1} = \dfrac{a + b - c}{a - b - c}.$

24. $\dfrac{.9x - .7y + 7.3}{13x - 15y + 17} = .2,$ $\dfrac{1.2x - .2y + 8.9}{13x - 15y + 17} = .3.$

25. $x + y = \dfrac{2(a^2 + b^2)}{a^2 - b^2},$ $x - y = \dfrac{4ab}{a^2 - b^2}.$

26. $\dfrac{x}{a + b} + \dfrac{y}{a - b} = \dfrac{1}{a - b},$ $\dfrac{x}{a + b} - \dfrac{y}{a - b} = \dfrac{1}{a + b}.$

27. $\dfrac{x}{a - b} - \dfrac{y}{a - c} = b - c,$ $\dfrac{a^2 - x}{b} + \dfrac{a^2 - y}{c} = b + c.$

28. $x = 1 + \sqrt{y}$,
$y = 4 - 3x + x^2$.

29. $2x - \sqrt{y} = 5$,
$(4x - 7)(x - 3) = y$.

30. $4\sqrt{x} - 3\sqrt{y} = 6$,
$3\sqrt{x} - 4\sqrt{y} = 1$.

31. $x\sqrt{a} - y\sqrt{b} = a + b$,
$x + y = 2\sqrt{a}$.

32. $x\sqrt{2} + y\sqrt{3} = 3\sqrt{3}$,
$x\sqrt{3} - y\sqrt{2} = 2\sqrt{2}$.

33. $4\sqrt{x+7} - 5\sqrt{y-7} = 7$,
$3\sqrt{x+7} - 7\sqrt{y-7} = 2$.

34. $\dfrac{7}{\sqrt{x}} + \dfrac{4}{\sqrt{y}} = 4$,
$\dfrac{1}{\sqrt{x}} + \dfrac{2}{\sqrt{y}} = 1$.

35. $\dfrac{8}{\sqrt{x-3}} - \dfrac{3}{\sqrt{y+3}} = 1$,
$\dfrac{4}{\sqrt{x-3}} + \dfrac{9}{\sqrt{y+3}} = 4$.

36. $(a - b)x + y = \dfrac{a+b+1}{a+b}$,
$(a-b)[x+(a+b)y] = a-b+1$.

37. $\dfrac{x+1}{3} - \dfrac{y+2}{4} = \dfrac{2(x-y)}{5}$,
$3(x-3) - 4(y-3) = 12(2y-x)$.

62. Solutions of problems involving two unknowns. The same principle of translation of the problem into algebraic symbols should be followed here as in the solution of problems leading to simple equations (p. 37).

PROBLEMS

1. The difference between two numbers is $3\frac{1}{3}$. Their sum is $9\frac{1}{9}$. What are the numbers?

2. What are the numbers whose sum is a and whose difference is b?

3. A man bought a pig and a cow for \$100. If he had given \$10 more for the pig and \$20 less for the cow, they would have cost him equal amounts. What did he pay for each?

4. Two baskets contain apples. There are 51 more in the first basket than in the second. But if there were 3 times as many in the first and 7 times as many in the second, there would be only 5 more in the first than in the second. How many apples are there in each basket?

5. A says to B, "Give me \$49 and we shall then have equal amounts." B replied, "If you give me \$49, I shall have 3 times as much as you. How much had each?

6. A man had a silver and a gold watch and two chains, the value of the chains being \$9 and \$25. The gold watch and the better chain are together twice and a half as valuable as the silver watch and cheaper chain. The gold watch and cheaper chain are worth \$2 more than the silver watch and the better chain. What is the value of each watch?

7. What fraction is changed into $\frac{1}{2}$ when both numerator and denominator are diminished by 7, and into its reciprocal when the numerator is increased by 12 and the denominator decreased by 12?

8. A man bought 2 carriage horses and 5 work horses, paying in all \$1200. If he had paid \$5 more for each work horse, a carriage horse would have been only $\frac{1}{4}$ more expensive than a work horse. How much did each cost?

9. A man's money at interest yields him \$540 yearly. If he had received $\frac{1}{2}$% more interest, he would have had \$60 more income. How much money has he at interest?

10. A man has two sums of money at interest, one at 4%, the other at 5%. Together they yield \$750. If both yielded 1% more interest, he would have \$165 more income. How large are the sums of money?

11. A man has two sums of money at interest, the first at 4%, the second at $3\frac{1}{2}$%. The first yields as much in 21 months as the second does in 18 months. If he should receive $\frac{1}{2}$% less from the first and $\frac{1}{2}$% more from the second, he would receive yearly \$7 more interest from both sums. What are the sums at interest?

12. What values have a mark and a ruble in our money if 38 rubles are worth 14 cents less than 75 marks, and if a dollar and a ruble together make $6\frac{1}{8}$ marks?

13. A chemist has two kinds of acid. He finds that 23 parts of one kind mixed with 47 parts of the other give an acid of $84\frac{5}{7}$% strength and that 43 parts of the first with 17 parts of the second give an $80\frac{5}{6}$% pure mixture. What per cent pure are the two acids?

14. Two cities are 30 miles apart. If A leaves one city 2 hours earlier than B leaves the other, they meet $2\frac{1}{2}$ hours after B starts. Had B started 2 hours earlier, they would have met 3 hours after he started. How many miles per hour do they walk?

15. The crown of Hiero of Syracuse, which was part gold and part silver, weighed 20 pounds, and lost $1\frac{1}{4}$ pounds when weighed in water. How much gold and how much silver did it contain if $19\frac{1}{4}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose one pound in water?

16. Two numbers which are written with the same two digits differ by 36. If we add to the lesser the sum of its tens digit and 4 times its units digit, we obtain 100. What are the numbers?

17. In a company of 14 persons, men and women, the men spend \$24 and the women spend an equal amount. If each man spends \$1 more than each woman, how many men and how many women are in the company?

63. Solution of linear equations in several variables. This process is performed as follows:

RULE. *Eliminate one variable from the equations taken in pairs, thus giving a system of one less equation than at first in one less variable.*

Continue the process until the value of one variable is found. The remaining variables may be found by substitution.

Special cases occur, as in the case of two variables, where an infinite number of solutions or no solutions exist. Where no solution exists one is led to a self-contradictory equation on application of the rule. See exercise 17, p. 48.

EXERCISES

Solve and check the following:

1. $$\begin{aligned} x + y + z &= 9, \\ x + 2y + 4z &= 15, \\ x + 3y + 9z &= 23. \end{aligned}$$

Solution:
$$\begin{array}{r} x + y + z = 9 \\ x + 2y + 4z = 15 \\ \hline y + 3z = 6 \end{array} \qquad \begin{array}{r} x + y + z = 9 \\ x + 3y + 9z = 23 \\ \hline 2y + 8z = 14 \\ y + 4z = 7 \end{array}$$

$$\begin{array}{r} y + 3z = 6 \\ y + 4z = 7 \\ \hline z = 1 \\ y + 3 = 6. \\ y = 3. \\ x + 3 + 1 = 9. \\ x = 5. \end{array}$$

Check: $5 + 9 + 9 = 23.$

2. $$\begin{aligned} x + y &= 37, \\ x + z &= 25, \\ y + z &= 22. \end{aligned}$$

3. $$\begin{aligned} x + y &= xy, \\ 2x + 2z &= xz, \\ 3z + 3y &= zy. \end{aligned}$$

HINT. Divide the equations by xy, xz, yz respectively.

4. $$\begin{aligned} x + y + z &= 17, \\ x + z - y &= 13, \\ x + z - 2y &= 7. \end{aligned}$$

5. $$\begin{aligned} x + y + z &= 36, \\ 4x &= 3y, \\ 2x &= 3z. \end{aligned}$$

6. $$\begin{aligned} 1.3x - 1.9y &= 1, \\ 1.7y - 1.1z &= 2, \\ 2.9x - 2.1z &= 3. \end{aligned}$$

7. $$\begin{aligned} 2x + 2y + z &= a, \\ 2y + 2z + x &= b, \\ 2z + 2x + y &= c. \end{aligned}$$

8. $$\begin{aligned} x + 2y &= 5, \\ y + 2z &= 8, \\ z + 2u &= 11, \\ u + 2x &= 6. \end{aligned}$$

9. $$\begin{aligned} x + y &= m, \\ y + z &= a, \\ z + u &= n, \\ u - x &= b. \end{aligned}$$

10. $$\begin{aligned} \frac{1}{y} + \frac{1}{z} &= 2a, \\ \frac{1}{x} + \frac{1}{z} &= 2b, \\ \frac{1}{x} + \frac{1}{y} &= 2c. \end{aligned}$$

11. $$\begin{aligned} \frac{xy}{x+y} &= \frac{1}{5}, \\ \frac{xz}{x+z} &= \frac{1}{6}, \\ \frac{yz}{y+z} &= \frac{1}{7}. \end{aligned}$$

12. $$\begin{aligned} \frac{xy}{4y-3x} &= 20, \\ \frac{xz}{2x-3z} &= 15, \\ \frac{yz}{4y-5z} &= 12. \end{aligned}$$

13. $$\begin{aligned} \frac{x+1}{y+1} &= 2, \\ \frac{y+2}{z+1} &= 4, \\ \frac{z+3}{x+1} &= \frac{1}{2}. \end{aligned}$$

14. $$\begin{aligned} 2\tfrac{1}{2}x &= y + z + 8, \\ 3\tfrac{1}{3}y &= x + z + 12, \\ 4\tfrac{1}{4}z &= x + y + 15. \end{aligned}$$

15. $$\begin{aligned} x + y &= 1\tfrac{1}{2}z + 8, \\ x + z &= 2\tfrac{2}{3}y - 14, \\ y + z &= 3\tfrac{3}{4}x - 32. \end{aligned}$$

16. $$\begin{aligned} x + 2y - z &= 4.6, \\ y + 2z - x &= 10.1, \\ z + 2x - y &= 5.7. \end{aligned}$$

17. $$\begin{aligned} x + 2y + 3z &= 15, \\ 3x + 5y + 7z &= 37, \\ 5x + 8y + 11z &= 59. \end{aligned}$$

18. $$\begin{aligned} 7x + 6y + 7z &= 100, \\ x - 2y + z &= 0, \\ 3x + y - 2z &= 0. \end{aligned}$$

19. $$\begin{aligned} (x+2)(2y+1) &= (2x+7)y, \\ (x-2)(3z+1) &= (x+3)(3z-1), \\ (y+1)(z+2) &= (y+3)(z+1). \end{aligned}$$

CHAPTER V

RATIO AND PROPORTION

64. Ratio. The **ratio** of one of two numbers to the other is the result of dividing one of them by the other.

The ratio of a to b is denoted by $a : b$ or by $\frac{a}{b}$.

The dividend in this implied division is called the **antecedent,** the divisor is called the **consequent.**

65. Proportion. Four numbers, a, b, c, d, are in **proportion** when the ratio of the first pair equals the ratio of the second pair.

This is denoted by $a : b = c : d$ or by $\frac{a}{b} = \frac{c}{d}$.

The letters a and d are called the **extremes,** b and c the **means,** of the proportion.

66. Theorems concerning proportion. If a, b, c, d are in proportion, that is, if

$$a : b = c : d \text{ or } \frac{a}{b} = \frac{c}{d}, \tag{I}$$

then

$$ad = bc, \tag{II}$$

$$b : a = d : c, \tag{III}$$

$$a : c = b : d, \tag{IV}$$

$$a + b : a = c + d : c, \tag{V}$$

$$a - b : a = c - d : c, \tag{VI}$$

$$a + b : a - b = c + d : c - d. \tag{VII}$$

Equation (III) is said to be derived from (I) by **inversion.**

Equation (IV) is said to be derived from (I) by **alternation.**

Equation (V) is said to be derived from (I) by **composition.**

Equation (VI) is said to be derived from (I) by **division.**

Equation (VII) is said to be derived from (I) by **composition and division.**

67. THEOREM. *If a number of ratios are equal, the sum of any number of antecedents is to any antecedent as the sum of the corresponding consequents is to the corresponding consequent.*

Let $$a : b = c : d = e : f = g : h,$$

or $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

To prove $$\frac{a + c + e}{g} = \frac{b + d + f}{h}.$$

If $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = r,$$

we have
$$a = br,$$
$$c = dr,$$
$$e = fr,$$
$$g = hr.$$

Divide the sum of the first three equations by the last and we get
$$\frac{a + c + e}{g} = \frac{b + d + f}{h}.$$

68. Mean proportion. The **mean proportional** between two numbers a and c is the number b, such that
$$a : b = b : c.$$

By (II), § 66, we see that $ac = b^2$.

EXERCISES

If $a : b = c : d$, prove that:

1. $a + b : \dfrac{a^2}{a + b} = c + d : \dfrac{c^2}{c + d}.$

Solution: By (V), § 66, $$\frac{a + b}{a} = \frac{c + d}{c}.$$

Squaring, we get $$\frac{(a + b)^2}{a^2} = \frac{(c + d)^2}{c^2},$$

or $$\frac{a + b}{\dfrac{a^2}{a + b}} = \frac{c + d}{\dfrac{c^2}{c + d}},$$

or $$a + b : \frac{a^2}{a + b} = c + d : \frac{c^2}{c + d}.$$

2. $a^2 : b^2 = c^2 : d^2$.

3. $a + b : c + d = a : \cancel{b}$ c

4. $ma : mb = nc : nd$.

5. $a^2 : \cancel{b^2} = a^2 + b^2 : c^2 + d^2$.

6. $a^2 + b^2 : \dfrac{a^3}{a+b} = c^2 + d^2 : \dfrac{c^3}{c+d}$.

7. $\sqrt{a^2 + c^2} : \sqrt{b^2 + d^2} = a : b$.

8. $ma + nb : ra + sb = mc + nd : rc + sd$.

9. $a + b + c + d : a - b + c - d = a + b - c - d : a - b - c + d$.

10. Find the mean proportional between $a^2 + c^2$ and $b^2 + d^2$.

11. Find the mean proportional between $a^2 + b^2 + c^2$ and $b^2 + c^2 + d^2$.

Solve the following for x:

12. $20 : 95 = x : 57$.

13. $8\,ab : x = bc : 1\tfrac{3}{4}\,ac$.

14. $x - ax : \sqrt{x} = \sqrt{x} : x$.

15. $1 - \sqrt{x} : 1 - 3\sqrt{x} = 1 : 4$.

16. $\dfrac{\sqrt{x+7} + \sqrt{x}}{\sqrt{x+7} - \sqrt{x}} = \dfrac{4 + \sqrt{x}}{4 - \sqrt{x}}$.

17. $\dfrac{a+b}{a-b} : \dfrac{a^2 - b^2}{ab} = x : \dfrac{a-b}{ac}$.

HINT. Use composition and division.

18. $\left(\dfrac{a^3 - b^3}{a - b} + ab\right) : \dfrac{a^3 + b^3}{a + b} - ab = (a + b)^2 : x$.

CHAPTER VI

IRRATIONAL NUMBERS AND RADICALS

69. Existence of irrational numbers. We have seen that in order to solve any linear equation or set of linear equations with rational coefficients we need to make use only of the operations of addition, subtraction, multiplication, and division. When, however, we attempt to solve the equation of the second degree, $x^2 = 2$, we find that there is no rational number that satisfies it.

ASSUMPTION. *A factor of one member of an identity between integers is also a factor of the other member.*

Thus let $2 \cdot a = b$, where a and b are integers. Then since 2 is a factor of the left-hand member, it must also be contained in b.

THEOREM. *No rational number satisfies the equation $x^2 = 2$.*

Suppose the rational number $\frac{a}{b}$ be a fraction reduced to its lowest terms which satisfies the equation. Then

$$\left(\frac{a}{b}\right)^2 = 2,$$

or

$$a^2 = 2\,b^2. \tag{1}$$

Thus, by the assumption, 2 is contained in a^2, and hence in a.

Suppose $$a = 2\,a'.$$

Then by (1) $$4\,a'^2 = 2\,b^2,$$

or $$2\,a'^2 = b^2,$$

that is, 2 must also be contained in b, which contradicts the hypothesis that $\frac{a}{b}$ is a fraction reduced to its lowest terms.

The fact that the equation $x^2 = 2$ has no rational solution is analogous to the geometrical fact that the hypotenuse of an isosceles right triangle is incommensurable with a leg.

70. The practical necessity for irrational numbers. For the practical purposes of the draughtsman, the surveyor, or the machinist, the introduction of this irrational number is superfluous, as no measuring rule can be made exact enough to distinguish between a length represented by a rational number and one that cannot be so represented. As the draughtsman does not use a mathematically perfect triangle, but one of rubber or wood, it is impossible to see in the fact of geometrical incommensurability just noted a *practical* demand from everyday life for the introduction of the irrational number. In fact the irrational number is a *mathematical* necessity, not a necessity for the laboratory or draughting room, as are the fraction and the negative number. We need irrational numbers because we cannot solve all quadratic equations without them, and the *practical* utility of those numbers comes only through the immense gain in mathematical power which they bring.

71. Extraction of square root of polynomials. This process, from which a method of extracting the square root of numbers is immediately deduced, may be performed as follows:

RULE. *Arrange the terms of the polynomial according to the powers of some letter.*

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, and add this quotient to the root and also to the trial divisor, thus forming the complete divisor.

Multiply the complete divisor by the last term of the root and subtract the product from the last remainder.

If terms of the given polynomial still remain, find the next term of the root by dividing the first term of the remainder by twice the first term of the root, form the complete divisor, and proceed as before until the desired number of terms of the root have been found.

EXERCISES

Extract the square root of the following:

1. $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

Solution:

$$\begin{array}{r|l}
 & a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4 \;\big|\; \underline{a^2 - ax + x^2} \\
 & \underline{a^4} \\
2a^2 - ax & -2a^3x + 3a^2x^2 - 2ax^3 + x^4 \\
 & \underline{-2a^3x + a^2x^2} \\
2a^2 - 2ax + x^2 & 2a^2x^2 - 2ax^3 + x^4 \\
 & \underline{2a^2x^2 - 2ax^3 + x^4}
\end{array}$$

2. $1 + x$.
3. $1 - x$.
4. $3x^2 - 2x + x^4 - 2x^3 + 1$.
5. $x^4 - 6x^3 + 13x^2 - 12x + 4$.
6. $x^4 + y^4 + 2x^3y - 2xy^3 - x^2y^2$.
7. $9x^4 - 12x^3 + 34x^2 - 20x + 25$.
8. $49a^4 - 42a^3b + 37a^2b^2 - 12ab^3 + 4b^4$.
9. $2ab - 2ac - 2bc + a^2 + b^2 + c^2$.
10. $u^4w^2 + v^4u^2 + w^4v^2 + 2u^3v^2w + 2v^3w^2u + 2w^3u^2v$.

72. Extraction of square root of numbers. We have the following

RULE. *Separate the number into periods of two figures each, beginning at the decimal point. Find the greatest number whose square is contained in the left-hand period. This is the first figure of the required root.*

Subtract its square from the first period, and to the remainder annex the next period of the number.

Divide this remainder, omitting the right-hand digit, by twice the root already found, and annex the quotient to both root and divisor, thus forming the complete divisor.

Multiply the complete divisor by the last digit of the root, subtract the result from the dividend, and annex to the remainder the next period for a new dividend.

Double the whole root now found for a new divisor and proceed as before until the desired number of digits in the root have been found.

In applying this rule it often happens that the product of the complete divisor and the last digit of the root is larger than the dividend. In such a case we must diminish the last figure of the root by unity until we obtain a product which is not greater than the dividend.

At any point in the process of extracting the square root of a number before the exact square root is found, the square of the result already obtained is less than the original number. If the last digit of the result be replaced by the next higher one, the square of this number is greater than the original number.

There are always two values of the square root of any number. Thus $\sqrt{4} = +2$ or -2, since $(+2)^2 = (-2)^2 = 4$. The positive root of any positive number or expression is called the principal root. When no sign is written before the radical, the principal root is assumed.

EXERCISES

Extract the square root of the following:

1. 2.0000.

Solution:

$$\begin{array}{r|l}
 & 2'.00'00'00' \;\underline{|1.414} \\
 & 1 \\
\hline
\underline{2.4} & 1.00 \\
 & 96 \\
\hline
\underline{281} & 400 \\
 & 281 \\
\hline
\underline{2.824} & 11900 \\
 & 11296 \\
\hline
 & 604
\end{array}$$

2. 95481.	**3.** 56169.	**4.** 3.
5. 877969.	**6.** 2949.5761.	**7.** 5.
8. 257049.	**9.** .00070128.	**10.** 99.
11. 69.8896.	**12.** .0009979281.	**13.** 12.
14. 49533444.	**15.** 9820.611801.	**16.** 160.

73. Approximation of irrational numbers. In the preceding process of extracting the square root of 2 we never can obtain a number whose square is exactly 2, for we have seen that such a number expressed as a rational (i.e. as a decimal) fraction does not exist. But as we proceed we get a number whose square differs less and less from 2.

Thus

$$\begin{aligned}
1.^2 &= 1, \text{ less than 2 by 1.} \\
1.4^2 &= 1.96, \text{ less than 2 by .04.} \\
1.41^2 &= 1.9881, \text{ less than 2 by .0119.} \\
1.414^2 &= 1.999396, \text{ less than 2 by .000604.}
\end{aligned}$$

Though we cannot say that 1.414 *is* the square root of 2, we may say that 1.414 is the square root of 2 correct to three decimal places, meaning that

$$(1.414)^2 < 2 < (1.415)^2.$$

74. Sequences. The exact value of the square root of most numbers, as, for instance, 2, 3, 5, cannot be found exactly in decimal form and so are usually expressed symbolically. By means of the process of extracting square root, however, we can find a number whose square is as near the given number as we may desire. We may, in fact, assert that the succession or sequence of numbers obtained by the process of extracting the square root of a number defines the square root of that number. Thus the sequence of numbers (1, 1.4, 1.41, 1.414, $\cdots$) defines the square root of 2.

75. Operations on irrational numbers. Just as we defined the laws of operation on the fraction and negative numbers (pp. 2–4), we should now define the meaning of the sum, difference, product, and quotient of the numbers defined by the sequence of numbers obtained by the square-root process. To define and explain completely the operations on irrational numbers is beyond the scope of this chapter. It turns out, however, that the number defined by a sequence is the limiting value of the rational numbers that constitute that sequence, that is, it is a value from which every number in the sequence beyond a certain point differs by as little as we please. We may, however, make the following statement regarding the multiplication of irrational numbers: In the sequence defining the square root of 2, namely, (1, 1.4, 1.41, 1.414, $\cdots$) we saw that we could obtain a number very nearly equal to 2 by multiplying 1.414 by itself. In general, *we multiply numbers defined by sequences by multiplying the elements of these sequences; the new sequence, consisting of the products, defines the product of the original numbers.*

Thus $\quad (1, 1.4, 1.41, 1.414, \cdots)(1, 1.4, 1.41, 1.414, \cdots)$

$$= (1, 1.96, 1.9881, 1.999396, \cdots).$$

The numbers in this sequence approach 2 as a limit, and hence the sequence may be said to represent 2.

76. Notation. We denote the square root of a (where a represents any number or expression) symbolically by $\sqrt{a}$, and assert that

$$\sqrt{a}\cdot\sqrt{a}=(\sqrt{a})^2=a,$$

or, more generally,

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}.$$

Similarly,

$$\sqrt{a}\div\sqrt{b}=\sqrt{a\div b}.$$

EXERCISES

1. Form five elements of a sequence defining $\sqrt{3}$.

2. Form five elements of a sequence defining $\sqrt{5}$.

3. Form five elements of a sequence defining $\sqrt{6}$.

4. Form, in accordance with the rule just given, four elements of the sequence $\sqrt{2}\cdot\sqrt{3}$. Compare the result with the elements obtained in Ex. 3.

5. Form similarly the first four elements of product $\sqrt{2}\cdot\sqrt{5}$ with the first four elements obtained by extracting the square root of 10.

77. Other irrational numbers. The cube root and higher roots of numbers could also be found by processes analogous to the method employed in finding the square root, but as they are almost never used practically, they will not be included here. It should be kept in mind, however, that by these processes sequences of numbers may be derived that define the various roots of numbers precisely as the sequences derived in the preceding paragraphs define the square root of numbers.

The nth root of any expression a is symbolized by $\sqrt[n]{a}$. Here n is sometimes called the **index of the radical.** The principle for the multiplication and division of radicals with any integral index is given by the following

ASSUMPTION. *The product (or quotient) of the nth root of two numbers is equal to the nth root of the product (or quotient) of the numbers.*

Symbolically expressed,

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{a\cdot b},$$

$$\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{a\div b}.$$

78. Reduction of a radical to its simplest form. A radical is in its **simplest form** when the expression under the sign is integral (§ 11) and contains no factor raised to a power which equals the index of the radical; in other words, when no factor can be removed from under the radical sign and still leave an integral expression. We may reduce a **quadratic** radical to its simplest form by the following

RULE. *If the expression under the radical sign is fractional, multiply both numerator and denominator by some expression that will make the denominator a perfect square.*

Factor the expression under the radical into two factors, one of which is the greatest square factor that it contains.

Take the square root of the factor that is a perfect square, and express the multiplication of the result by the remaining factor under the radical sign.

If the radical is of the nth index, the denominator must be made a perfect nth power, and any factor that is to be taken from under the radical sign must also be a perfect nth power.

EXERCISES

Reduce to simplest form:

1. $\sqrt{\frac{12}{5}}$.

Solution: $$\sqrt{\frac{12}{5}} = \sqrt{\frac{12 \cdot 5}{25}} = \sqrt{\frac{4 \cdot 15}{25}} = \frac{2}{5}\sqrt{15}.$$

2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{32}$.
4. $\sqrt{\frac{9}{8}}$.
5. $\sqrt{27}$.
6. $\sqrt{\frac{5}{14}}$.
7. $\sqrt{243}$.
8. $\sqrt[3]{250}$.
9. $\sqrt{\frac{1}{9} + 4}$.
10. $\sqrt{\frac{1}{3} - \frac{1}{4}}$.
11. $8\sqrt{75}$.
12. $\frac{4}{3}\sqrt{27 b^2}$.
13. $\frac{5}{8}\sqrt{80 x^3 y^4}$.
14. $\sqrt{\frac{1}{10} + \frac{1}{25}}$.
15. $\sqrt{\frac{25}{9} + \frac{9}{4}}$.
16. $\frac{12}{5}\sqrt{1 + \frac{1}{24}}$.
17. $\sqrt{\frac{1}{.75}}$.
18. $\sqrt{\frac{5x}{9}}$.
19. $\sqrt{\frac{25 b}{y^2}}$.
20. $\sqrt{\frac{5 a^5}{6 x^5}}$.
21. $b\sqrt{\frac{x^2}{b}}$.
22. $\frac{a^2}{b}\sqrt[4]{\frac{b^5 x}{a^9 y}}$.
23. $\sqrt{\frac{6 x^3}{8}}$.
24. $8\sqrt{\frac{27 a^3}{36 x^2}}$.
25. $8\sqrt{\frac{7 a}{16 x^2}}$.

26. $\sqrt{x^3 - 2x^2y + xy^2}$.

27. $\sqrt{5x^3 - 20x^2 + 20x}$.

28. $\sqrt{\dfrac{a^3 - 2a^2 + a}{ax^2 + bx^2}}$.

29. $\sqrt{\dfrac{2x^3 - 12x^2 + 18x}{50y - 20y^2 + 2y^3}}$.

30. $\sqrt{\dfrac{2a^3 - 8a^2 + 8a}{8x - 8x^2 + 2x^3}}$.

31. $\sqrt{\dfrac{a^3 + a^2b - ab^2 - b^3}{9(a - b)}}$.

79. Addition and subtraction of radicals. Radicals that are of the same index and have the same expression under the radical sign are **similar**. Only similar radicals can be united into one term by addition and subtraction. We add radical expressions by the following

RULE. *Reduce the radicals to be added to their simplest form. Add the coefficients of similar radicals and prefix this sum as the coefficient of the corresponding radical in the result.*

A rule precisely similar is followed in subtracting radical expressions.

EXERCISES

Add the following:

1. $\sqrt{27}$, $\sqrt{48}$, and $\sqrt{75}$.

Solution:

$$\begin{aligned} \sqrt{27} &= \sqrt{9 \cdot 3} = 3\sqrt{3} \\ \sqrt{48} &= \sqrt{16 \cdot 3} = 4\sqrt{3} \\ \sqrt{75} &= \sqrt{25 \cdot 3} = 5\sqrt{3} \\ \text{Sum} &= 12\sqrt{3} \end{aligned}$$

2. $\sqrt{3} + 2\sqrt{3}$.

3. $8\sqrt{7} - 3\sqrt{7}$.

4. $a\sqrt{x} - b\sqrt{x}$.

5. $a + 2\sqrt{a} + 3\sqrt[3]{a} + 2\sqrt{16a} - \sqrt[3]{27a}$.

6. $3\sqrt{8} + 4\sqrt{32} - 5\sqrt{50} - 7\sqrt{72} + 6\sqrt{98}$.

7. $8\sqrt{a} + 5\sqrt{x} - 7\sqrt{a} + 4\sqrt{a} - 6\sqrt{x} - 3\sqrt{a}$.

8. $7\sqrt{4x} + 4\sqrt{9x} + 3\sqrt{45x} - 5\sqrt{36x} - 2\sqrt{80x}$.

9. $\sqrt{a - b} + \sqrt{16a - 16b} + \sqrt{ax^2 - bx^2} - \sqrt{9(a - b)}$.

10. $4\sqrt{a^2x} - 3\sqrt{b^2x} + 2\sqrt{c^2x} + \sqrt{d^2x} - 2\sqrt{(b + d)^2x}$.

11. $6\sqrt{x} + 3\sqrt{2x} - 5\sqrt{3x} - 2\sqrt{4x} + \sqrt{12x} - \sqrt{18x}$.

80. Multiplication and division of radicals. For these processes we have the following

RULE. *Follow the usual laws of operation* (§ 10), *using also the assumption of* § 77.

Reduce each term of the result to its simplest form.

The operations of this section are limited to the case where the radicals are of the same index. Radicals of different indices as $\sqrt{3}$ and $\sqrt[3]{2}$ must first be reduced to the same index. See § 87.

EXERCISES

1. Multiply $\sqrt{2}-\sqrt{3}$ by $\sqrt{2}-\sqrt{8}$.

Solution:

$$\begin{array}{rl} & \sqrt{2}-\sqrt{3} \\ & \underline{\sqrt{2}-\sqrt{8}\qquad\qquad\quad} \\ & 2-\sqrt{6}-\sqrt{16}+\sqrt{24} \\ = & 2-\sqrt{6}-4+2\sqrt{6} \\ = & -2+\sqrt{6}. \end{array}$$

2. Divide $\dfrac{x+y}{\sqrt[3]{xy}}$ by $\sqrt[3]{x}+\sqrt[3]{y}$.

Solution:

$$\frac{\dfrac{x+y}{\sqrt[3]{xy}}}{\sqrt[3]{x}+\sqrt[3]{y}} = \frac{(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})(\sqrt[3]{x}+\sqrt[3]{y})^*}{\sqrt[3]{xy}(\sqrt[3]{x}+\sqrt[3]{y})}$$

$$= \frac{\sqrt[3]{x^2}}{\sqrt[3]{xy}} - \frac{\sqrt[3]{xy}}{\sqrt[3]{xy}} + \frac{\sqrt[3]{y^2}}{\sqrt[3]{xy}}$$

$$= \sqrt[3]{\frac{x}{y}} - 1 + \sqrt[3]{\frac{y}{x}}.$$

Carry out the indicated operations and simplify:

3. $\sqrt{10}\cdot\sqrt{5}$.
4. $\sqrt[3]{3}\cdot\sqrt[3]{18}$.
5. $\sqrt{28}\cdot\sqrt{7}$.
6. $\sqrt{\frac{2}{3}}\div\sqrt{\frac{3}{2}}$.
7. $(a-b\sqrt{x})^2$.
8. $\sqrt{\frac{24}{35}}\cdot\sqrt{\frac{10}{21}}$.
9. $(\sqrt{7}-\sqrt{3})(\sqrt{3}-\sqrt{2})$.
10. $(-1+\sqrt{3})^2$.
11. $(5\sqrt{3}+\sqrt{6})(5\sqrt{2}-2)$.
12. $\sqrt{14c}\cdot\sqrt{70c}$.

* Since $a^3+b^3=(a+b)(a^2-ab+b^2)$, if $a=\sqrt[3]{x}$, $b=\sqrt[3]{y}$ we have
$x+y=(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})$.

13. $(a+b-\sqrt{ab})(\sqrt{a}+\sqrt{b})$.

14. $(8+3\sqrt{5})(2-\sqrt{5})$.

15. $\sqrt{\sqrt{x}+\sqrt{y}}\cdot\sqrt{\sqrt{x}-\sqrt{y}}$.

16. $\sqrt{6+2\sqrt{5}}\cdot\sqrt{6-2\sqrt{5}}$.

17. $\sqrt[3]{x+\sqrt{x^2-1}}\cdot\sqrt[3]{x-\sqrt{x^2-1}}$.

18. $(x^2+y^2)\div(x\sqrt[3]{y}+y\sqrt[3]{x})$.

19. $\dfrac{a}{x}\div\sqrt{ax}$.

20. $(4\sqrt{a}-\sqrt{3x})(\sqrt{a}+2\sqrt{3x})$.

21. $\sqrt{a}\cdot\sqrt{\dfrac{5x}{4a}}$.

22. $a\div\sqrt{\dfrac{a}{b}}$.

23. $\dfrac{7}{\sqrt{8}}\div\sqrt{\dfrac{7}{8}}$.

24. $\left(\sqrt{\dfrac{a-x}{x-b}}-\sqrt{\dfrac{x-b}{a-x}}\right)^2$.

25. $\left(\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}\right)^2$.

26. $\sqrt{a^2-b^2}\cdot\sqrt{\dfrac{5a+5b}{ax^2-bx^2}}$.

27. $\dfrac{x^4-y^4}{\sqrt[3]{xy}}\div(x\sqrt[3]{x}-y\sqrt[3]{y})$.

28. $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)\div\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{y}}\right)$.

29. $(2\sqrt{6}-\sqrt{12}-\sqrt{24}+\sqrt{48})\sqrt{2}$.

30. $(3\sqrt{8}+\sqrt{18}+\sqrt{50}-2\sqrt{72})\sqrt{2}$.

31. $(5\sqrt{24}-4\sqrt{32}+3\sqrt{50}-3\sqrt{54})\sqrt{3}$.

32. $(\sqrt{9x+5}+3\sqrt{x})(\sqrt{9x+5}-3\sqrt{x})$.

33. $[(\sqrt{7}+\sqrt{3}+\sqrt{10})(\sqrt{7}+\sqrt{3}-\sqrt{10})]^2$.

34. $(2\sqrt{30}-3\sqrt{5}+5\sqrt{3})(\sqrt{8}+\sqrt{3}-\sqrt{5})$.

35. $(2\sqrt{5}+\sqrt{8}-\sqrt{12})(\frac{1}{3}\sqrt{30}-\frac{2}{3}\sqrt{3}+\sqrt{2})$.

36. $\left(\sqrt{\dfrac{a+1}{2}}+\sqrt{\dfrac{a-1}{2}}\right)\left(\sqrt{\dfrac{a+1}{2}}-\sqrt{\dfrac{a-1}{2}}\right)$.

37. Find the value of $\frac{1}{4}\sqrt{24}-\sqrt{\frac{2}{3}}+2\sqrt{3-\sqrt{5}}\cdot\sqrt{3+\sqrt{5}}$ to three decimal places.

81. Rationalization. The process of rendering the irrational numerator (or denominator) of a fractional expression rational without altering the value of the fraction is called the **rationalization** of the numerator (or denominator) of the fraction.

This is usually accomplished by multiplying both numerator and denominator of the given fraction by a properly chosen radical expression called the **rationalizing factor.**

The principles in accordance with which this rationalizing factor is selected are the following:

PRINCIPLE I. *Since* $(a+b)(a-b)=a^2-b^2$, *the rationalizing factor of* $\sqrt{x}\pm\sqrt{y}$ *is* $\sqrt{x}\mp\sqrt{y}$.

PRINCIPLE II. *Since* $(a^2-ab+b^2)(a+b)=a^3+b^3$, *the rationalizing factor of* $\sqrt[3]{x}+\sqrt[3]{y}$ *is* $\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$, *and conversely.*

Since $(a^2+ab+b^2)(a-b)=a^3-b^3$, *the rationalizing factor of* $\sqrt[3]{x}-\sqrt[3]{y}$ *is* $\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}$, *and conversely.*

EXERCISES

Rationalize the denominators of the following:

1. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

Solution: By Principle I the factor which will render the denominator rational is $\sqrt{a}+\sqrt{x}$.

Thus $$\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}=\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}\cdot\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}}=\frac{a+x+2\sqrt{ax}}{a-x}.$$

2. $\dfrac{1}{2+\sqrt{2}+\sqrt{3}}$.

Solution: This problem requires a twofold rationalization.

$$\begin{aligned}\frac{1}{2+\sqrt{2}+\sqrt{3}}&=\frac{(2+\sqrt{2})-\sqrt{3}}{[(2+\sqrt{2})+\sqrt{3}][(2+\sqrt{2})-\sqrt{3}]}\\&=\frac{2+\sqrt{2}-\sqrt{3}}{(2+\sqrt{2})^2-3}=\frac{2+\sqrt{2}-\sqrt{3}}{2+4\sqrt{2}+2-3}\\&=\frac{2+\sqrt{2}+\sqrt{3}}{1+4\sqrt{2}}=\frac{(2+\sqrt{2}+\sqrt{3})(1-4\sqrt{2})}{(1+4\sqrt{2})(1-4\sqrt{2})}\\&=\frac{1+\sqrt{2}+\sqrt{3}-4\sqrt{2}-8-4\sqrt{6}}{1-16\cdot 2}\\&=\frac{7+3\sqrt{2}+\sqrt{3}-4\sqrt{6}}{31}.\end{aligned}$$

3. $\dfrac{1}{\sqrt[3]{2}-\sqrt[3]{3}}$.

Solution: $$\begin{aligned}\frac{1}{\sqrt[3]{2}-\sqrt[3]{3}}&=\frac{1}{\sqrt[3]{2}-\sqrt[3]{3}}\cdot\frac{\sqrt[3]{2^2}+\sqrt[3]{6}+\sqrt[3]{3^2}}{\sqrt[3]{2^2}+\sqrt[3]{6}+\sqrt[3]{3^2}}=\frac{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}{2-3}\\&=-(\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}).\end{aligned}$$

4. $\dfrac{1}{2+\sqrt{3}}$. 5. $\dfrac{7-\sqrt{5}}{3+\sqrt{5}}$. 6. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

7. $\dfrac{2}{\sqrt[3]{3}+\sqrt[3]{2}}$. 8. $\dfrac{8}{\sqrt[3]{2}-\sqrt[3]{4}}$. 9. $\dfrac{5\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}}$.

10. $\sqrt{\dfrac{a+\sqrt{x}}{a-\sqrt{x}}}$. 11. $\sqrt{\dfrac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}}$.

12. $\dfrac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. 13. $\dfrac{1+3\sqrt{2}-2\sqrt{3}}{\sqrt{6}+\sqrt{3}+\sqrt{2}}$.

14. $\dfrac{2\sqrt{15}}{\sqrt{3}+\sqrt{5}+2\sqrt{2}}$. 15. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

16. $\dfrac{2}{\sqrt{a+1}+\sqrt{a-1}}$. 17. $\dfrac{\sqrt{6}-\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}}$.

18. Show that $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}=-.10\cdots$.

19. Show that $\dfrac{3\sqrt{8}-2\sqrt{7}}{\sqrt{8}-\sqrt{7}}=2.51\cdots$.

20. $\dfrac{\sqrt{(1+a)(1+b)}-\sqrt{(1-a)(1-b)}}{\sqrt{(1+a)(1+b)}+\sqrt{(1-a)(1-b)}}$.

21. Show that $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}=.158$.

82. Solution of equations involving radicals. We prove the important

THEOREM. *When an equation in x is multiplied by an expression in x, the resulting equation has, in general, solutions which the first one did not possess.*

Let
$$A=0$$
represent an equation containing x which is satisfied by the values $x=a, b, \cdots n$. Let B be an expression which vanishes when $x=\alpha, \beta, \cdots \nu$. Then the expression
$$A\cdot B=0$$
is satisfied not only when $x=a, b, \cdots, n$, but also when
$$x=\alpha, \beta, \cdots, \nu.$$

EXAMPLE. The equation
$$x - 2 = 0$$
has $x = 2$ for its only solution, while the equation
$$(x-2)(x-3) = 0$$
has in addition the solution $x = 3$.

If in the course of a problem it is necessary to multiply an equation by any expression involving the variable, the solutions of the resulting equation must be substituted in the first one to ascertain if any solutions have been introduced which did not satisfy the original equation. Solutions which have been introduced in the process of solving an equation, but which do not satisfy the original equation, are called **extraneous solutions.**

It may be shown in a similar way that raising the equation in x,
$$A = B,$$
to any power introduces extraneous solutions.

EXERCISES

1. Solve $\sqrt{x+19} + \sqrt{x+10} = 9.$

Solution:
$$\begin{aligned}
\sqrt{x+19} &= 9 - \sqrt{x+10}.\\
x + 19 &= 81 + x + 10 - 18\sqrt{x+10}.\\
-72 &= -18\sqrt{x+10}.\\
4 &= \sqrt{x+10}.\\
16 &= x + 10.\\
x &= 6.
\end{aligned}$$

Check: $\sqrt{6+19} + \sqrt{6+10} \doteq 5 + 4 = 9.$

2. Solve $\sqrt{x+19} - \sqrt{x+10} = -9.$
$$\sqrt{x+19} = -\left(9 - \sqrt{x+10}\right).$$

Simplifying, we get $\quad x = 6.$

Check: $\sqrt{6+19} - \sqrt{6+10} = +5 - 4 = 1 \neq 9.$

Thus our result satisfies only the equation which was introduced in the course of solving the problem, and is extraneous. The original equation has no solution.

Solve and check, noting all extraneous solutions:

3. $\sqrt{3x} - 1 = 5.$
4. $\sqrt{\frac{1}{2}x} - 3 = 2.$
5. $\sqrt{2x} + \sqrt{3x} = 1.$
6. $\sqrt{x} + \sqrt{3x} = 2.$
7. $\sqrt[3]{5x-7} = \sqrt[3]{4x+3}.$
8. $5\sqrt{x} - 7 = 3\sqrt{x} - 1.$

9. $\sqrt{x+1}+\sqrt{x+2}=3.$

10. $\sqrt{13+4\sqrt{x-1}}=5.$

11. $2\sqrt{x}-\sqrt{2x}=2+\sqrt{2}.$

12. $\sqrt{x+4}+\sqrt{x+1}=1.$

13. $\sqrt{37-7\sqrt{5x-4}}=4.$

14. $\sqrt[4]{x^2-7x+19}=\sqrt{x-3}.$

15. $7+\sqrt{x^2-11x+4}=x.$

16. $8+\sqrt{(x-10)(x-5)}=x.$

17. $x-\sqrt{ax(1+x)+1-x}=1.$

18. $\sqrt{2(x+1)}+\sqrt{2x+15}=13.$

19. $\frac{2}{5}(7\sqrt{x}+5)-5=\frac{3}{2}(3\sqrt{x}-1).$

20. $2\sqrt{3}+3\sqrt{2x}=3\sqrt{2}+2\sqrt{3x}.$

21. $\dfrac{9}{5+\sqrt{x}}=\dfrac{4}{8-\sqrt{x}}.$

22. $\dfrac{5+\sqrt{x}}{5-\sqrt{x}}=4.$

23. $\sqrt{7x+2}=\dfrac{5x+6}{\sqrt{7x+2}}.$

24. $\dfrac{a-\sqrt{bx}}{a+\sqrt{bx}}=\dfrac{2b-3\sqrt{ax}}{2b+3\sqrt{ax}}.$

25. $\sqrt{2x-1}=\dfrac{2(x-3)}{\sqrt{2x-10}}.$

26. $\sqrt{9x+10}=\dfrac{6x+10}{\sqrt{4x+9}}.$

27. $\dfrac{\sqrt{1-x}}{1+\sqrt{1-x}}=\dfrac{\sqrt{1-x}}{1-\sqrt{1-x}}.$

28. $\sqrt{a-x}+\sqrt{b-x}=\dfrac{b}{\sqrt{b-x}}.$

29. $x-ax:\sqrt{x}=\sqrt{x}:x.$

30. $2\sqrt{2x+2}+\sqrt{x+2}=\dfrac{12x+4}{\sqrt{8x+8}}.$

31. $\dfrac{1+2\sqrt{3x-5}}{1+3\sqrt{3x-5}}=\dfrac{11+2\sqrt{3x-5}}{11+5\sqrt{3x-5}}.$

32. $\sqrt{9x+7}+\sqrt{4x+1}=\sqrt{25x+14}.$

33. $\sqrt{x+15}+\sqrt{x-24}-\sqrt{x-13}=\sqrt{x}.$

34. $\sqrt{x-7}+\sqrt{x-2}-\sqrt{x-10}=\sqrt{x+5}.$

35. $(\sqrt{x}-7)(\sqrt{x}-3)=(\sqrt{x}-6)(\sqrt{x}-5).$

36. $(a+\sqrt{x})\sqrt{x}:(b-\sqrt{x})\sqrt{x}=a+1:b-1.$

37. $(4\sqrt{x}-7):(5\sqrt{x}-6)=(\sqrt{x}-7):(\sqrt{x}-6).$

38. $\left(\sqrt{a\sqrt{b}}-\sqrt{b\sqrt{a}}\right)\sqrt{x}=a\sqrt{b\sqrt{x}}-b\sqrt{a\sqrt{x}}.$

CHAPTER VII

THEORY OF INDICES

83. Negative exponents. We have already seen (§ 16) that

$$a^n \cdot a^m = a^{n+m} \tag{1}$$

when n and m are positive integers. We now assume that this law still holds when one or both of the numbers m and n are negative or fractional.

If we let $$a^{-m} = \frac{1}{a^m},$$

then $$\frac{a^n}{a^m} = a^n\left(\frac{1}{a^m}\right) = a^n \cdot a^{-m} = a^{n-m},$$

since the law (1) holds when n and m are any integers. This notation may be expressed verbally as follows:

PRINCIPLE. *A factor of numerator or denominator of a fraction may be changed from the numerator to the denominator, or vice versa, if the sign of its exponent be changed.*

84. Fractional exponents. Since (p. 57) $\sqrt{a} \cdot \sqrt{a} = a$, it is natural to devise a notation for $\sqrt{a}$ suggested by the law (1).

If we let $$\sqrt{a} = a^{\frac{1}{2}},$$
we have $$\sqrt{a} \cdot \sqrt{a} = a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

Furthermore, if we let $\sqrt[n]{a} = a^{\frac{1}{n}}$,
it would be consistent with law (1) to write

$$\left(a^{\frac{1}{n}}\right)^2 = a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} = a^{\frac{1}{n}+\frac{1}{n}} = a^{\frac{2}{n}}.$$

This notation we shall assume in general. Thus

$$\left(\sqrt[n]{a}\right)^m = \left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}.$$

With the adoption of this notation we can attach a meaning to any real number with any rational number for its exponent. This notation may be expressed verbally in the following

PRINCIPLE. *The numerator of a fractional exponent indicates a power, the denominator a root.*

85. Further assumptions. The operation of multiplication is subject to the following laws of exponents:

I. *Commutative law of rational exponents:*

$$(a^q)^r = a^{q \cdot r} = a^{r \cdot q} = (a^r)^q.$$

II. *Associative law of rational exponents:*

$$(a^{qr})^s = a^{qr \cdot s} = a^{q \cdot rs} = (a^q)^{rs}.$$

The laws of operation (§ 10) defined for integral values of the symbols we also assume when the symbols are expressions with rational exponents.

86. THEOREM. $a^r b^r = (ab)^r$, *where r is any rational number as* $\frac{p}{q}$.

We raise both sides of the equation $a^r b^r = (ab)^r$ to the qth power separately and show that the results are equal.

Since $\qquad r = \frac{p}{q},$

$$[(ab)^r]^q = \left[(ab)^{\frac{p}{q}}\right]^q = (ab)^p = a^p b^p.$$

Also $\qquad (a^r b^r)^q = \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q = \underbrace{\left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)\left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right) \cdots \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)}_{q \text{ terms}}$

$$= \underbrace{\left(a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots a^{\frac{p}{q}}\right)}_{q \text{ terms}} \underbrace{\left(b^{\frac{p}{q}} \cdot b^{\frac{p}{q}} \cdots b^{\frac{p}{q}}\right)}_{q \text{ terms}}$$

$$= \left(a^{\frac{p}{q}}\right)^q \left(b^{\frac{p}{q}}\right)^q = a^p b^p.$$

Thus $\qquad (a^r b^r)^q = [(ab)^r]^q.$

Extracting the qth root and taking the principal root, we obtain

$$a^r b^r = (ab)^r.$$

EXERCISES

1. Express in simplest form with positive exponents:

(a) $\dfrac{36\,a^{-2}b^{-1}c^{-\frac{1}{2}}}{9\,a^{2}b^{-2}c^{-\frac{3}{2}}}$.

Solution: By Principle, § 83, $\dfrac{36\,a^{-2}b^{-1}c^{-\frac{1}{2}}}{9\,a^{2}b^{-2}c^{-\frac{3}{2}}}$

$$= \frac{4\,b^{-1}b^{+2}c^{-\frac{1}{2}}c^{+\frac{3}{2}}}{a^2\cdot a^2}$$

$$= \frac{4\,bc}{a^4}.$$

(b) $\dfrac{a^3}{a^{-3}}$.

(c) $\dfrac{b^{-4}}{b^{-9}}$.

(d) $\dfrac{a^{-2}b^3}{x^4y^{-5}}$.

(e) $\dfrac{x^{\frac{1}{3}}y^{\frac{2}{3}}z^{\frac{1}{2}}}{\sqrt[3]{x^2y}\cdot\sqrt{z}}$.

(f) $\dfrac{\sqrt[6]{(x^{12}y^6)^5}}{(x^{\frac{3}{5}}y^{\frac{2}{5}})^{-\frac{4}{5}}}$.

(g) $\dfrac{21\,x^{-1}y^5z^{-3}}{35\,x^{-2}y^6z^{-4}}$.

(h) $\dfrac{3\,a^{-1}b^{-2}}{4\,x^{-2}y^{-4}}\cdot\dfrac{6\,a^2x^{-1}}{5\,b^{-1}c^2}$.

(i) $\dfrac{ab^{\frac{1}{2}}c\,\sqrt{a}\,b\,\sqrt[3]{c}}{\sqrt[3]{a^2}\,b^{-\frac{1}{3}}\,a^{\frac{1}{3}}b^2c^{\frac{4}{3}}}$.

(j) $x^{\frac{5}{6}}y^{\frac{1}{4}}z^{\frac{1}{2}}\sqrt{x^{-\frac{1}{4}}y^{-1}z^{-1}}$.

(k) $\dfrac{4\,x^{-n}y^{-3}}{5\,a^{-4}b^{-m}}\cdot\dfrac{15\,a^{-2}b^{3-m}}{14\,x^ny^{n-3}}$.

2. Arrange in order of magnitude the following:

(a) $\sqrt{\frac{4}{3}},\ \sqrt[3]{\frac{3}{2}},\ \sqrt[4]{\frac{7}{4}}$.

Solution: We first ask, Is $(\frac{4}{3})^{\frac{1}{2}} > (\frac{3}{2})^{\frac{1}{3}}$?

Raise both numbers to the sixth power.

We obtain $(\frac{4}{3})^3$ and $(\frac{3}{2})^2$,

or $\frac{64}{27}$ and $\frac{9}{4}$,

or $2\frac{10}{27}$ and $2\frac{1}{4}$.

Thus $\sqrt{\frac{4}{3}} > \sqrt[3]{\frac{3}{2}}$.

Now compare $(\frac{4}{3})^{\frac{1}{2}}$ and $(\frac{7}{4})^{\frac{1}{4}}$.

Raise both numbers to the fourth power.

We obtain $(\frac{4}{3})^2$ and $\frac{7}{4}$,

or. $1\frac{7}{9}$ and $1\frac{3}{4}$.

Thus $\sqrt{\frac{4}{3}} > \sqrt[4]{\frac{7}{4}}$.

Now compare $(\frac{3}{2})^{\frac{1}{3}}$ and $(\frac{7}{4})^{\frac{1}{4}}$.

Raise both numbers to the twelfth power.

We obtain $(\frac{3}{2})^4$ and $(\frac{7}{4})^3$,

or $\frac{81}{16}$ and $\frac{343}{64}$,

or $5\frac{1}{16}$ and $5\frac{23}{64}$.

Thus $\sqrt[4]{\frac{7}{4}} > \sqrt[3]{\frac{3}{2}}$.

The order of magnitude is then $\sqrt{\frac{4}{3}}$, $\sqrt[4]{\frac{7}{4}}$, $\sqrt[3]{\frac{3}{2}}$.

(b) $\sqrt[3]{2}$, $\sqrt[4]{\frac{4}{3}}$, $\sqrt[4]{\frac{7}{8}}$.

(c) $\sqrt[3]{25}$, $\sqrt{8}$, $\sqrt[4]{75}$.

(d) $\sqrt[4]{\frac{21}{25}}$, $\sqrt{.1}$.

(e) $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[5]{15}$.

3. Perform the indicated operations.

(a) $\sqrt[3]{2} \cdot \sqrt{3}$.

Solution:
$$\sqrt[3]{2} \cdot \sqrt{3} = 2^{\frac{1}{3}} \cdot 3^{\frac{1}{2}} = 2^{\frac{2}{6}} \cdot 3^{\frac{3}{6}}$$
$$= (2^2 \cdot 3^3)^{\frac{1}{6}} = \sqrt[6]{4 \cdot 27} = \sqrt[6]{108}.$$

(b) $\sqrt[3]{4} \cdot \sqrt{5}$.

(c) $\dfrac{\sqrt[4]{2} \cdot \sqrt{6}}{\sqrt[3]{3}}$.

(d) $\dfrac{\sqrt[5]{2} \cdot \sqrt[3]{3}}{\sqrt[3]{5}}$.

(e) $\dfrac{\sqrt{5} \cdot \sqrt[3]{6}}{\sqrt[4]{3} \cdot \sqrt{2}}$.

87. Operations with radical polynomials. These operations follow the rules for the same operations previously given, provided the assumptions and principles of §§ 83–86 are observed.

EXERCISES

1. Divide $x^{\frac{3}{5}} - y^{\frac{3}{5}}$ by $\sqrt[5]{x} - \sqrt[5]{y}$.

2. Extract the square root of $4x - 12x^{\frac{1}{2}}y^{\frac{1}{3}} + 9y^{\frac{2}{3}} + 32x^{\frac{1}{2}} - 48y^{\frac{1}{3}} + 64$.

3. Simplify $\dfrac{3x + 3x^{-1} - 6}{x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} - x^{-\frac{3}{2}}}$.

4. Divide $\sqrt{\frac{4}{3}\sqrt{a^3}} - \sqrt{\frac{3}{4}\sqrt{a^3}}$ by $3\sqrt{\frac{3}{a}}$.

5. Extract the square root of $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} - 2$.

6. Multiply $-3x^{-5} + 2\dfrac{b^{-1}}{x^4}$ by $\dfrac{-2}{x^3} - \dfrac{3x^{-4}}{b^{-1}}$.

7. Extract the square root of
$$\frac{4}{49}x^2y^{-2} - \frac{15}{2}yx^{-1} + \frac{9}{16}y^2x^{-2} - \frac{20}{7}xy^{-1} + 25\frac{3}{7}.$$

8. Multiply $\sqrt{x^7} - x^3 + x^{\frac{5}{2}} - \dfrac{1}{x^{-2}} + \sqrt{x^3} - x + \sqrt{x} - 1$ by $\sqrt{x} + 1$.

QUADRATICS AND BEYOND

CHAPTER VIII

QUADRATIC EQUATIONS

88. Definition. An equation that contains the second but *no higher power of the variable* is called a **quadratic equation.** The most general form of the quadratic equation in one variable is

$$ax^2 + bx + c = 0, \tag{1}$$

where we shall always assume a, b, and c to represent rational numbers, and where $a \neq 0$. Every quadratic equation in x can be brought to this form by transposing and simplifying.

89. Solution of quadratic equations. The **solution** of a quadratic equation consists in finding its roots, that is, the numbers (or expressions involving the coefficients in case the coefficients are literal) which satisfy the equation.

The common method of solving a quadratic equation consists in bringing the member of the equation that involves the variable into the form of a perfect square, i.e. into the form

$$x^2 + 2Ax + A^2.$$

For example, let us solve

$$x^2 + 2x - 8 = 0.$$

Transpose 8, $\qquad x^2 + 2x = 8.$

If now we add 1 to both sides of the equation, the left-hand member will be a perfect square,

$$x^2 + 2x + 1 = 9.$$

Express as a square, $(x+1)^2 = 9.$

Extract the square root, $x + 1 = \pm 3.$

Transpose, $x = -4 \text{ or } 2.$

Both -4 and 2 satisfy the equation, as we see on substituting them for x. Thus

$$(-4)^2 + 2(-4) - 8 = 0,$$

and

$$2^2 + 2 \cdot 2 - 8 = 0.$$

Consider now the general case.

Let us solve $ax^2 + bx + c = 0.$

Transpose c, $ax^2 + bx = -c.$

Divide by a, $x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Add $\left(\frac{b}{2a}\right)^2$ to both members to make the left-hand member a perfect square,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2}.$$

Express as a square, $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$

Extract the square root,

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Transpose, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$* (1)

The roots are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

That the equation can have no other roots appears from § 96.

* This expression for the roots,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

may be used as a formula for the solution of a quadratic equation.

Thus to solve the equation $2x^2 - 3x + 6 = 0$

we may substitute in the formula $a = 2$, $b = -3$, $c = 6$, and obtain

$$x = \frac{3 \pm \sqrt{9 + 48}}{4} = \frac{3 \pm \sqrt{57}}{4}.$$

Thus $x_1 = \frac{3 + \sqrt{57}}{4}, \quad x_2 = \frac{3 - \sqrt{57}}{4}.$

One should verify the fact that both $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ satisfy (1) and are consequently roots of the equation. They are, in general, distinct from each other. For particular values of the coefficients to be noted later (§ 98) the roots may be equal or complex (i.e. of form $\alpha+\beta\sqrt{-1}$, where α and β are ordinary rational or irrational numbers).

We may sum up the process of solving a quadratic equation in the following

RULE. *Write the expression in the form* $ax^2+bx+c=0$.

Transpose the term not involving x *to the right-hand side of the equation.*

Divide both sides of the equation by the coefficient of x^2.

Add to both members the square of one half of the coefficient of x, *thus making the left-hand member a perfect square.*

Rewrite the equation, expressing the left-hand member as the square of a binomial and the right-hand member in its simplest form.

Extract the square root of both members of the equation, not omitting the $\pm$ *sign in the right-hand member.*

Transpose the constant term, leaving x *alone on the left-hand side of the equation. The two values obtained on the right-hand side by taking the* $+$ *and* $-$ *signs separately are the roots sought.*

Check by substituting the solutions in the original equation, which should then reduce to an identity.

90. Pure quadratics. A quadratic equation in which the coefficient of the term in x is zero is often called a **pure quadratic.** Its solution is found precisely as in the general case, excepting that we do not need to complete the square. Thus let us solve

$$ax^2+c=0.$$

Transpose c, $$ax^2=-c.$$

Divide by a, $$x^2=-\frac{c}{a}.$$

Extract the square root, $\quad x = \pm\sqrt{-\frac{c}{a}}.$

The roots are $x_1 = +\sqrt{-\frac{c}{a}},\ x_2 = -\sqrt{-\frac{c}{a}}.$

EXERCISES

Solve and check the following:

1. $3x^2 - 6x - 10 = 0.$

Solution: Transpose 10, $\quad 3x^2 - 6x = 10.$

Divide by 3, $\quad x^2 - 2x = \frac{10}{3}.$

Add the square of $\frac{1}{2}$ the coefficient of x, i.e. 1, to both sides,

$$x^2 - 2x + 1 = \tfrac{10}{3} + 1 = \tfrac{13}{3}.$$

Express as a square, $\quad (x-1)^2 = \frac{13}{3}.$

Extract the square root, $\quad x - 1 = \pm\sqrt{\frac{13}{3}}.$

$$x = 1 \pm \sqrt{\tfrac{13}{3}}.$$

Check: $3\left(1 \pm \sqrt{\frac{13}{3}}\right)^2 - 6\left(1 \pm \sqrt{\frac{13}{3}}\right) - 10 = 0.$

$$3 \pm 6\sqrt{\frac{13}{3}} + \frac{39}{3} - 6 \mp \frac{6\sqrt{13}}{3} - 10 = 0.$$

2. $8x^2 + 2x - 3 = 0.$

Solution: Transpose 3, $\quad 8x^2 + 2x = 3.$

Divide by 8, $\quad x^2 + \frac{1}{4}x = \frac{3}{8}.$

Add the square of $\frac{1}{2}$ the coefficient of x, i.e. $\frac{1}{64}$, to both sides,

$$x^2 + \tfrac{1}{4}x + \tfrac{1}{64} = \tfrac{3}{8} + \tfrac{1}{64} = \tfrac{25}{64}.$$

Express as a square, $\quad (x + \frac{1}{8})^2 = \frac{25}{64}.$

Extract the square root, $\quad x + \frac{1}{8} = \pm\frac{5}{8}.$

$$x = \pm\tfrac{5}{8} - \tfrac{1}{8}$$
$$= -\tfrac{3}{4} \text{ or } \tfrac{1}{2}.$$

Check: $8\cdot(\frac{1}{2})^2 + 2\cdot\frac{1}{2} - 3 = \frac{8}{4} + 1 - 3 = 0.$

$$8(-\tfrac{3}{4})^2 + 2(-\tfrac{3}{4}) - 3 = \tfrac{72}{16} - \tfrac{6}{4} - 3 = \tfrac{9}{2} - \tfrac{3}{2} - 3 = \tfrac{9}{2} - \tfrac{9}{2} = 0.$$

3. $x^2 - ax = 0.$

4. $x^2 = 169.$

5. $x^2 - \frac{1}{2}x = \frac{1}{2}.$

6. $\frac{5}{7}x^2 = 560.$

7. $x^2 + x - 1 = 0.$

8. $19x^2 = 5491.$

9. $3x^2 - 7x = 16.$

10. $x^2 = .074529.$

11. $3x^2 + 11 = 5x.$

12. $x^2 - 1\frac{1}{2}x = 1.$

13. $x^2 + x - 56 = 0.$

14. $20x^2 + x = 12.$

15. $7x^2 + 9x = 100.$

16. $6x^2 + 5x = 56.$

17. $14x^2 - 33 = 71x.$

18. $5x^2 + 13 = 14x.$

19. $x^2 - 8x + 15 = 0.$

20. $91x^2 - 2x = 45.$

21. $x^2 + 2x - 63 = 0.$

22. $x^2 - 6x + 16 = 0.$

23. $x^2 - 10x + 32 = 0.$

24. $6x^2 + 26\frac{1}{4} = 25\frac{1}{2}x.$

25. $6x^2 - 13x + 6 = 0.$

26. $15x^2 + 527 = 178x.$

27. $2x^2 + 15.9 = 13.6x.$

28. $(x-1)^2 = a(x^2-1).$

29. $a^2(b-x)^2 = b^2(a-x)^2.$

30. $13x^2 - 19 = 7x^2 + 5.$

31. $ax^2 - (a^2+1)x + a = 0.$

32. $(a-x)(x-b) = -ab.$

33. $14x^2 + 45.5x = -36.26.$

34. $a^2(a-x)^2 = b^2(b-x)^2.$

35. $(a-x)^2 + (x-b)^2 = a^2 + b^2.$

36. $(a-x)(x-b) = (a-x)(c-x).$

37. $\dfrac{ax^2}{b} = \dfrac{c}{d}.$

38. $2x + \dfrac{1}{x} = 3.$

39. $\dfrac{15x}{2} = \dfrac{810}{3x}.$

40. $x^2 + \dfrac{x}{7} = 50.$

41. $\dfrac{2x}{3} = \dfrac{1050}{7x}.$

42. $\dfrac{a+x}{b+x} + \dfrac{b+x}{a+x} = \dfrac{5}{2}.$

43. $\dfrac{x+11}{x+3} = \dfrac{2x+1}{x+5}.$

44. $\dfrac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3.$

45. $\dfrac{ax+b}{bx+a} = \dfrac{mx-n}{nx-m}.$

46. $\dfrac{21}{x} - \dfrac{10}{x-2} - \dfrac{4}{x-3} = 0.$

47. $\dfrac{ax^2 - bx + c}{lx^2 - mx + n} = \dfrac{c}{n}.$

48. $\dfrac{5x-7}{9} + \dfrac{14}{2x-3} = x - 1.$

49. $\dfrac{x-2}{3x+14} = \dfrac{3(8-x)}{28-x}.$

50. $\dfrac{(a-x)^3 + (x-b)^3}{(a-x)-(x-b)} = \dfrac{a^3-b^3}{a+b}.$

51. $\dfrac{7}{2x-3} + \dfrac{5}{x-1} = 12.$

52. $\dfrac{16-x}{4} - \dfrac{2(x-11)}{x-6} = \dfrac{x-4}{12}.$

53. $\dfrac{5+x}{3-x} - \dfrac{8-3x}{x} = \dfrac{2x}{x-2}.$

54. $\dfrac{2x-1}{x-2} + \dfrac{3x+1}{x-3} = \dfrac{5x-14}{x-4}.$

55. $\left(\dfrac{a-x}{x-b}\right)^2 = 8\left(\dfrac{a-x}{x-b}\right) - 15.$

56. $\dfrac{5x-1}{9} + \dfrac{3x-1}{5} = \dfrac{2}{x} + x - 1$

57. $a^2 - x^2 = (a-x)(b+c-x).$

58. $\dfrac{1}{1+\sqrt{1-x}} + \dfrac{1}{1-\sqrt{1-x}} = \dfrac{2x}{9}.$

59. $\dfrac{2x+2}{18} + \dfrac{12}{x+4} = \dfrac{x-4}{4} + \dfrac{x-2}{6}.$

60. $(x-a+b)(x-a+c) = (a-b)^2 - x^2.$

91. Solution of quadratic equations by factoring. When the left-hand member of an equation can be factored readily, this is the most convenient method of solution. It also illustrates very clearly the meaning and property of the roots of the equation.

EXAMPLE. Solve $x^2 + 2x - 15 = 0.$

Factor the left-hand member, $(x + 5)(x - 3) = 0.$

The object in solving an equation is to find numbers that substituted for the variable satisfy the equation. But since zero multiplied by any number is zero (p. 3), any value of x which causes *one* factor of an expression to vanish makes the whole expression vanish. If in this case $x = 3$, our equation in factored form becomes

$$(3 + 5)(3 - 3) = (3 + 5) \cdot 0 = 0$$

and is satisfied. If we let $x = -5$, the other factor becomes zero, and the equation reduces to the identity

$$(5 - 5)(5 - 3) = 0(5 - 3) = 0.$$

Thus the numbers 3 and -5 are solutions of the equation.

92. Solution of an equation by factoring. We have immediately the

RULE. *Transpose all the terms to the left-hand member of the equation.*

Factor that member into linear factors.

The values of the variable that make the factors vanish are roots of the equation.

EXERCISES

Solve and check the following:

1. $6x^2 + x = 15.$
2. $6x^2 + 7x = 3.$
3. $5x^2 - x - 6 = 0.$
4. $5x^2 - 17x + 6 = 0.$
5. $13x^2 - 38x = 3.$
6. $2x^2 - 5x - 25 = 0.$
7. $x^2 - 40x + 111 = 0.$
8. $13x^2 - 40x + 3 = 0.$
9. $x^2 - 18x - 208 = 0.$
10. $3x^2 - 26x + 35 = 0.$
11. $x^2 - 3ax - 4a^2 = 0.$
12. $(x - a)^2 - (x - b)^2 = 0.$
13. $(x^2 - 1) + (x - 1)^2 = 0.$
14. $(3x - 5)^2 - (9x + 1)^2 = 0.$
15. $(2x - 1)(x + 2) + (x - 1)(x - 2) = -4.$
16. $(7x - 1)(x + 3) - (4x - 3)(x - 1) = 24.$

17. $(3x+1)(x+1)-(4x+3)(x-1)=-2.$

18. $(x-a)(4ax-b)+(x-b)(4ax-b)=0.$

19. $3x-4\sqrt{x-7}=2(x+2).$

Solution: Transpose, $\quad x-4=4\sqrt{x-7}.$

Square, $\quad x^2-8x+16=16x-112.$

Transpose, $\quad x^2-24x+128=0.$

Factor, $\quad (x-16)(x-8)=0.$

The roots are $\quad x=16,\quad x=8.$

Check: $3\cdot 16-4\sqrt{16-7}-2(16+2)=48-12-32-4=0.$

$$3\cdot 8-4\sqrt{8-7}-2(8+2)=24-4-20=0.$$

In the following examples, as always, the quadratic equation should be solved by factoring when possible. Recourse to the longer but sure method of completing the square is always available.

When an equation is cleared of fractions or squared in the process of bringing it into quadratic form (1), § 88, extraneous solutions may be introduced. The results should be verified in every case and extraneous solutions rejected.

20. $a+\sqrt{a^2-x^2}=x.$

21. $\sqrt{x+5}=x-1.$

22. $2x-\sqrt{2x-1}=x+2.$

23. $1-6x+\sqrt{5(x+4)}=0.$

24. $\sqrt{11-x}+\sqrt{x-2}=3.$

HINT. Square twice.

25. $\sqrt{2x+1-2\sqrt{2x+3}}=1.$

26. $\sqrt{a(x-b)}+\sqrt{b(x-a)}=x.$

27. $\sqrt{x+3}+\sqrt{2x-3}=6.$

28. $2\sqrt{3+x}-4\sqrt{3-x}=\sqrt{60}.$

29. $\sqrt{1+ax}-\sqrt{1-ax}=x.$

30. $\sqrt{5x-1}-\sqrt{8-2x}=\sqrt{x-1}.$

31. $\sqrt{x+7-\sqrt{5x-2}}=3.$

32. $x-10=\frac{2}{3}(x-1)-\sqrt{2x-1}.$

33. $\sqrt{a^2-x}+\sqrt{b^2+x}=a+b.$

34. $\sqrt{4-x}+\sqrt{5-x}=\sqrt{9-2x}.$

35. $(a^2-b^2)(x^2+1)=2(a^2+b^2)x.$

36. $x+2a\sqrt{2(a^2+b^2)-x}=3a^2+b^2.$

37. $x\sqrt{x-2}+2\sqrt{x+2}=\sqrt{x^3+8}.$

38. $\sqrt{\dfrac{a+x}{b+x}}+\sqrt{\dfrac{a-x}{b-x}}=2\sqrt{\dfrac{a}{b}}.$

39. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{a}{x}.$

HINT. Rationalize.

40. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}=\dfrac{2\sqrt{x}}{\sqrt{a}+\sqrt{x}}-\dfrac{(x+a)^2}{a(x-a)}.$

41. $\dfrac{\sqrt{a-x}+\sqrt{x-b}}{\sqrt{a-x}-\sqrt{x-b}}=\sqrt{\dfrac{a-x}{x-b}}.$

42. $\sqrt{x}+\dfrac{\sqrt{b}-\sqrt{a}}{\sqrt{b}}=\dfrac{1}{\sqrt{x}}+\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}}.$

43. $\dfrac{2a-(1+a^2)x}{1+a^2-2ax}=\dfrac{2b+(1+b^2)x}{1+b^2+2bx}.$

44. $2\sqrt{x+4}-3\sqrt{2x-3}=\dfrac{4}{\sqrt{x+4}}.$

45. $(a-x)^2+(b-x)^2=\frac{5}{2}(a-x)(b-x).$

46. $abx^2-(a+b)(ab+1)x+(ab+1)^2=0.$

47. $\sqrt{2x-2}+\sqrt{3x+7}=\sqrt{2x+11}+\sqrt{3x-8}.$

48. $\sqrt{(a+x)(x+b)}+\sqrt{(a-x)(x-b)}=2\sqrt{ax}.$

49. $2\sqrt{2a+b+2x}-\sqrt{10a+b-6x}=\sqrt{10a+9b-6x}.$

93. Quadratic form. Any equation is in **quadratic form** if it may be written as a trinomial consisting of a constant term and two terms involving the variable (or an expression which may be considered as the variable), the exponent in one term being twice that in the other. By the **constant term** is meant the term not containing the variable.

Thus $x-8\sqrt{x}+13=0$, $x^{-\frac{4}{3}}+x^{-\frac{2}{3}}-3=0$, $a^2x^{-2n}-(a+b)x^{-n}+b^2=0$, $x^2-2x-3-\sqrt{x^2-2x-3}+17=0$ are all in quadratic form. In the last the whole expression x^2-2x-3 is taken as the variable.

It is usually convenient to replace by a single letter the lower power of the variable or expression with respect to which the equation is in quadratic form.

EXERCISES

Solve and check the following:

1. $x-8\sqrt{x}+15=0.$ (1)

Solution: Let $\sqrt{x}=y.$

Then $x=y^2.$

Substituting, (1) becomes $y^2-8y+15=0.$

Factor, $(y-5)(y-3)=0.$

The roots are $y=5,\ y=3.$

Thus $\sqrt{x}=5,\ \sqrt{x}=3,$

or $x=25,\ x=9.$

Check: $25-8\cdot5+15=0;\ 9-8\cdot3+15=0.$

2. $x^{-\frac{4}{3}} - 5x^{-\frac{2}{3}} + 4 = 0.$ (1)

Solution: Let $x^{-\frac{2}{3}} = y.$

Then $x^{-\frac{4}{3}} = y^2.$

Substituting, (1) becomes $y^2 - 5y + 4 = 0.$

Factor, $(y-4)(y-1) = 0.$

The roots are $y = 4, \ y = 1.$

Since $$x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{x^2}},$$

we have $$\frac{1}{\sqrt[3]{x^2}} = 4, \quad \frac{1}{\sqrt[3]{x^2}} = 1.$$

Thus $$\sqrt[3]{x^2} = \frac{1}{4}; \quad x^2 = \frac{1}{64}; \quad x = \pm\frac{1}{8};$$

and $$\sqrt[3]{x^2} = \frac{1}{1}; \quad x^2 = 1; \quad x = \pm\sqrt{1} = \pm 1.$$

3. $2\sqrt{x^2 - 2x - 3} + x^2 - 2x = 6.$

Solution: Add -3 to both members and rearrange terms,

$$x^2 - 2x - 3 + 2\sqrt{x^2 - 2x - 3} - 3 = 0. \qquad (1)$$

Let $\sqrt{x^2 - 2x - 3} = y.$

Then $x^2 - 2x - 3 = y^2.$

Substituting, (1) becomes $y^2 + 2y - 3 = 0.$

Factor, $(y+3)(y-1) = 0.$

The roots are $y = -3, \ y = 1.$

Hence $y^2 = x^2 - 2x - 3 = 9,$

or $x^2 - 2x + 1 = 13.$

Extract the square root, $x - 1 = \pm\sqrt{13}.$

The roots are $x = 1 \pm \sqrt{13}.$

Also $x^2 - 2x - 3 = 1,$

or $x^2 - 2x + 1 = 5.$

Extract the square root, $x - 1 = \pm\sqrt{5}.$

The roots are $x = 1 \pm \sqrt{5}.$

4. $x^3 - 1 = 0.$

5. $x^3 - 8 = 0.$

6. $x^6 - 1 = 0.$

7. $x^{\frac{3}{2}} + 8x^{\frac{1}{2}} = 9x.$

Hint. Divide by $\sqrt{x}$. This factor corresponds to the root $x = 0$.

8. $x - 6x^{-1} = 1.$

9. $4x^{\frac{2}{3}} + 5x^{\frac{1}{3}} - 1 = 0.$

10. $10x^4 - 21 = x^2.$

11. $2x^{\frac{1}{3}} - 3x^{\frac{2}{3}} + x = 0.$

12. $3\sqrt[3]{x^2} + 6\sqrt[3]{x} = 4.$

13. $ax^{2p} + bx^p + c = 0.$

14. $3x^4 - 7x^2 - 6 = 0.$

15. $4x^6 - 14x^3 + 6 = 0.$

16. $x^4 - 13x^2 + 36 = 0.$

17. $x - 12\sqrt{x} + 11 = 0.$

18. $x^3 + 4x^{\frac{1}{3}} = 16\frac{1}{4}x\sqrt[3]{x^2}.$

19. $\sqrt[4]{x^3} - 2\sqrt{x} + x = 0.$

20. $8x^{-6} + 999x^{-3} = 125.$

21. $7\sqrt[7]{x^5} + 5x\sqrt[7]{x^3} = 66.$

22. $2(\sqrt{x} - 3)^2 - 3 = \sqrt{x}.$

23. $(x^2 - 10)(x^2 - 3) = 78.$

24. $(x^2 - 1)^2 + (x^2 + 1) = 2.$

25. $(\sqrt[3]{x} - 1)^2 + \sqrt[3]{x^2} = \sqrt[3]{x}.$

26. $x^{\frac{5}{3}} + x^{\frac{1}{3}} = (2^3 + 2^{-3})x\sqrt[5]{x^{\frac{1}{3}}}.$

27. $(\sqrt[4]{x} - 3)(\sqrt[4]{x} - 4) = 12.$

28. $(x + a)^{\frac{1}{2}} + (x + b)^{\frac{1}{2}} = (a - b)^{\frac{1}{2}}.$

29. $(2x^2 - 3)^2 - (x^2 + 4)^2 = 7.$

30. $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3.$

31. $(2x + 3)^{\frac{1}{2}} + (2x + 3)^{-\frac{1}{2}} = 6.$

32. $8(8x - 5)^3 + 5(5 - 8x)^6 = 85.$

33. $x^4 - 4(a+b)x^2 + 16(a-b)^2 = 0.$

34. $4x^2 + 12x\sqrt{1 + x} = 27(1 + x).$

35. $(2x^2 - 3x + 1)^2 = 22x^2 - 33x + 1.$

36. $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) = 1.$

37. $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}.$

38. $2\sqrt[3]{(x - 4)^2} + 4(x - 4)^{-\frac{2}{3}} = 9.$

39. $x^2 - \dfrac{3}{x^2} = 2.$

40. $\dfrac{x - 4}{2 + x} = x - 8.$

41. $x^{-3} + \dfrac{1}{x\sqrt{x}} = 2.$

42. $a = x^2 + b + \dfrac{b^2}{x^2}.$

43. $\sqrt{x} - 8 = \dfrac{7}{\sqrt{x} - 2}.$

44. $1 + \dfrac{x}{2} - \dfrac{x^2}{2(1 + \sqrt{1 + x})^2} = 3.$

45. $x^{\frac{5}{3}} + \dfrac{41\sqrt[3]{x}}{x} = \dfrac{97}{\sqrt[3]{x^2}} + x^{\frac{4}{3}}.$

46. $\dfrac{2x}{(x - 1)^2} + \dfrac{(x - 1)^2}{2x} = 4.$

47. $(x - 1)^2 + 4(x - 1) - 5 = 0.$

HINT. Let $y = \dfrac{2x}{(x-1)^2}$; then $\dfrac{1}{y} = \dfrac{(x-1)^2}{2x}.$

48. $(x^2 + 2)^{\frac{3}{2}} + \dfrac{3}{\sqrt{x^2 + 2}} = 4x^2 + 8.$

94. Problems solvable by quadratic equations. The principle of translating the problem into algebraic symbols, explained in § 55, should be observed here. The result should be verified in every case. It may happen that the problem implies restrictions that are not expressed in the equation to which the problem leads. In this case some of the solutions of the equation may not be consistent with the problem; for instance, when the

variable stands for a number of men fractional solutions should be rejected. If only such results are obtained, the problem is self-contradictory. Often negative solutions should be rejected, as when the result indicates a negative number of digits in a number. Imaginary or complex (p. 72) results in general mean that the conditions of the problem cannot be realized.

PROBLEMS

1. The product of $\frac{1}{4}$ and $\frac{1}{5}$ of a certain number is 500. What is the number?

2. There are two numbers one of which is less than 100 by as much as the other exceeds it. Their product is 9831. What are the numbers?

3. The sum of the square roots of two numbers is $\sqrt{74}$. One of the numbers is less than 37 by as much as the other exceeds it. What are the numbers?

4. A man sells oranges for $\frac{1}{12}$ as many cents apiece as he has oranges. He sells out for $3. How many had he?

5. If the perimeter of a square is 100 feet, how long is its diagonal?

6. A man sells goods and makes as much per cent as $\frac{1}{5}$ the number of dollars in the buying price. He made $245. What was the buying price?

7. Two bodies A and B move on the sides of a right angle. A is now 123 feet from the vertex and is moving away from it at the rate of 239 feet per second. B is 239 feet from the vertex and moves toward it at a rate of 123 feet per second. At what time (past or future) are they 850 feet apart?

8. What is the number twice whose square exceeds itself by 190?

9. What numbers have a sum equal to 53 and a product equal to 612?

10. The sum of the squares of two numbers whose difference is 12 is found to be 1130. What are the numbers?

11. By what number must one increase each factor of $24 \cdot 20$ so that the product shall be 540 greater?

12. What numbers have a quotient 4 and a product 900?

13. Two numbers are in the ratio of $4:5$. Increase each by 15 and the difference of their squares is 999. What are the numbers?

14. If $4\frac{1}{2}$ is divided by a certain number, the same result is obtained as if the number had been subtracted from $4\frac{1}{2}$. What is the number?

15. Separate 900 into two parts such that the sum of their reciprocal values is the reciprocal of 221.

16. The denominator of a fraction is greater by 4 than the numerator. Decrease the numerator by 3 and increase the denominator by the same, and the resulting fraction is half as great as the original one. What is the original fraction?

17. The numerator and denominator of a fraction are together equal to 100. Increase the numerator by 18 and decrease the denominator by 16, and the fraction is doubled. What is the fraction?

18. A number consists of two digits whose sum is 10. Reverse the order of the digits and multiply the resulting number by the original one, and the result is 2944. What is the number?

19. The sum of two numbers is 200. The square root of one increased by the other is 44. What are the numbers?

20. The difference of two numbers is 10, and the difference of their cubes is 20530. What are the numbers?

21. Around a rectangular flower bed which is 3 yards by 4 yards there extends a border of turf which is everywhere of equal breadth and whose area is ten times the area of the bed. How wide is it?

22. Two bicyclists travel toward each other, starting at the same time from places 51 miles apart. One goes at the rate of 9 miles an hour. The number of miles per hour gone by the other is greater by 5 than the number of hours before they meet. How far does each travel before they meet?

23. A printed page has 15 lines more than the average number of letters per line. If the number of lines is increased by 15, the number of letters per line must be decreased by 10 in order that the amount of matter on the two pages may be the same. How many letters are there on the page?

24. A merchant buys goods for a certain sum. The cost of handling them was 5% of their cost price. He sells for $504, gaining as much per cent as $\frac{1}{20}$ the cost price was in dollars. What was the cost price?

25. A man had $8000 at interest. He increased his capital by $100 at the end of each year, apart from his interest. At the beginning of the third year he had $8982.80. What per cent interest did his money draw?

26. Two men A and B can dig a trench in 20 days. It would take A 9 days longer to dig it alone than it would B. How long would it take B alone?

27. A cistern is emptied by two pipes in 6 hours. How long would it take each pipe to do the work if the first can do it in 5 hours less time than the second?

28. A party procures lunch at a restaurant for $15. If there had been 5 less in the party, each member would have paid 15 cents more without affecting the amount of the entire bill. How many were in the party?

29. A party pays \$12 for accommodations. Had there been 4 more in the party, and if each person had paid 25 cents less, the bill would have been \$15. How many were in the party?

30. A grocer sells his stock of butter for \$15. If he had had 5 pounds less in stock, he would have been obliged to charge 10 cents more a pound to realize the same amount. How many pounds had he in stock?

31. A man buys lemons for \$2. If he had received for that money 50 more lemons, they would have cost him 2 cents less apiece. What was the price of each lemon?

32. It took a number of men as many days to dig a ditch as there were men. If there had been 6 more men, the work would have been done in 8 days. How many men were there?

95. Theorems regarding quadratic equations. In this and the following sections we prove several theorems concerning quadratic equations. Similar theorems are later proved in general for equations of higher degree.

THEOREM. *If α is a root of the equation*

$$ax^2 + bx + c = 0, \tag{1}$$

then $x - \alpha$ is a factor of its left-hand member, and conversely.

The fact that α is a root of the equation is equivalent to the assertion of the truth of the identity

$$a\alpha^2 + b\alpha + c \equiv 0,$$

by definition of the root of an equation (§ 53).

Divide $ax^2 + bx + c$ by $x - \alpha$ as follows:

$$\begin{array}{l|l|l}
x - \alpha & ax^2 + bx + c & ax + (b + a\alpha) \\
\hline
 & ax^2 - a\alpha x & \\
\cline{2-2}
 & (b + a\alpha)x + c & \\
 & (b + a\alpha)x - a\alpha^2 - b\alpha & \\
\cline{2-2}
 & \qquad\qquad a\alpha^2 + b\alpha + c &
\end{array}$$

Since the remainder vanishes by hypothesis, $ax^2 + bx + c$ is exactly divisible by $x - \alpha$.

Conversely, we have already seen (p. 3) that if $x - \alpha$ is a factor of an equation, α is a root, since replacing x by α would make that factor vanish.

EXERCISES

Form equations of which the following are roots.

1. 2, 6.

Solution: Since 2 and 6 are roots, $(x-2)$ and $(x-6)$ are factors, and the equation having these as factors is

$$(x-2)(x-6)=x^2-8x+12=0.$$

2. $1, -1$.
3. $-3, -4$.
4. $\sqrt{2}, -\sqrt{2}$.
5. $-2, -6$.
6. $-4\sqrt{2}, \sqrt{32}$.
7. $\sqrt{27}, -3\sqrt{3}$.
8. $2+\sqrt{3}, 2-\sqrt{3}$.
9. $a+\sqrt{b}, a-\sqrt{b}$.

96. THEOREM. *A quadratic equation has only two roots.*

Given the equation $ax^2+bx+c=0$ with the roots α and β, to prove that the equation has no other root, as γ, distinct from α and β.

Since α and β are roots of the equation, $x-\alpha$ and $x-\beta$ are factors. Thus our equation may be written in the form

$$a(x-\alpha)(x-\beta)=0. \tag{1}$$

If now γ is a root, it must satisfy (1), i.e.

$$a(\gamma-\alpha)(\gamma-\beta)=0.$$

But in order that any product of numerical factors should vanish one of the factors must vanish. Thus either $a=0$, or $\gamma-\alpha=0$, or $\gamma-\beta=0$. But, by hypothesis, $\gamma\neq\alpha$ and $\gamma\neq\beta$, so the last two factors cannot vanish. Thus $a=0$. This would, however, reduce our equation to a linear equation, which is contrary to our hypothesis that the equation is quadratic. Thus the assumption that we have three distinct roots leads to a contradiction.

COROLLARY I. *If a quadratic equation is satisfied by more than two distinct values of the variable, then each of the coefficients vanishes identically.*

The above proof shows that the coefficient of x^2 must vanish. In the same way it can be shown that $b=c=0$.

COROLLARY II. *If two quadratic equations have the same value for more than two values of the variable, then their coefficients are identical.*

Let $$ax^2+bx+c=a'x^2+b'x+c'$$

for more than two values of x. Transpose, and we obtain

$$(a-a')x^2+(b-b')x+c-c'=0.$$

We have then a quadratic equation satisfied by more than two values of x. Thus by Corollary I each of its coefficients must vanish. Thus $a'=a$, $b'=b$, $c'=c$.

This theorem taken with § 95 is equivalent to the statement that a quadratic equation can be factored in one and only one way. Thus if

$$ax^2 + bx + c = a(x - \alpha)(x - \beta),$$

we cannot find other numbers γ and δ distinct from α and β such that

$$ax^2 + bx + c = a(x - \gamma)(x - \delta),$$

for then the equation would have roots distinct from α and β.

97. Theorem. *If the equation*

$$x^2 + bx + c = 0, \tag{1}$$

where b and c are integers, has rational roots, those roots must be integers.

For suppose $\frac{p}{q}$ to be a rational fraction reduced to its lowest terms and a root of (1).

Then $$\frac{p^2}{q^2} + \frac{bp}{q} + c = 0,$$

or $$p^2 + bpq + cq^2 = 0,$$

which gives $$p^2 = -q(bp + cq).$$

Thus some factor of q must be contained in p (§ 69), which contradicts the hypothesis that the fraction $\frac{p}{q}$ is already reduced to its lowest terms.

98. Nature of the roots of a quadratic equation. The equation

$$ax^2 + bx + c = 0 \tag{1}$$

has as roots (§ 89) $$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{2}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{3}$$

These expressions afford an immediate arithmetic means of determining the nature of the roots of the given equation when a, b, and c have numerical values and $a \neq 0$. In fact an inspection of the value of $b^2 - 4ac$ is sufficient to determine the nature of the roots of (1).

I. *When $b^2 - 4ac$ is negative, the roots are imaginary* (§ 89).

II. *When $b^2 - 4ac = 0$, the roots are real and equal. In this case $x_1 = x_2 = -\frac{b}{2a}$.*

III. *When $b^2 - 4ac$ is positive, the roots are real and distinct.*

IV. *When $b^2 - 4ac$ is positive and a perfect square, the roots are real, distinct, and rational.*

In case IV, if $$b^2 - 4\,ac = \Delta,$$

$$x_1 = \frac{-b + \sqrt{\Delta}}{2\,a}; \quad x_2 = \frac{-b - \sqrt{\Delta}}{2\,a}.$$

The converses of these four cases are also true. For instance, if the roots of (1) are imaginary, from (2) and (3) it is clear that $b^2 - 4\,ac$ must be negative.

The expression $\Delta = b^2 - 4\,ac$ is called the **discriminant** of the equation $ax^2 + bx + c = 0$.

EXERCISES

1. Determine the nature of the roots of the following equations without solving.

(a) $3x^2 - 4x - 1 = 0$.

Solution: $\Delta = (-4)^2 - 4 \cdot 3 \cdot (-1) = 16 + 12 = 28$ and is then positive. Thus by III the roots are real and distinct.

(b) $3x^2 - 7x + 6 = 0$.

(c) $6x^2 - x - 1 = 0$.

(d) $3x^2 + 4x + 1 = 0$.

(e) $x^2 - 4x + 1 = 0$.

(f) $2x^2 - 6x - 9 = 0$.

(g) $2x^2 - 4x - 2 = 0$.

(h) $4x^2 + 12x + 9 = 0$.

(i) $2x^2 + 6x - 4 = 0$.

(j) $4x^2 - 28x + 49 = 0$.

(k) $4x^2 + 12x + 5 = 0$.

2. Determine real values of k so that the roots of the following equations may be equal. Check the result.

(a) $(2 + k)x^2 + 2\,kx + 1 = 0$.

Solution: Here $2 + k = a, \quad 2\,k = b, \quad 1 = c.$

Thus
$$\Delta = b^2 - 4\,ac = 4\,k^2 - 4 \cdot (k+2) \cdot 1$$
$$= 4\,k^2 - 4\,k - 8.$$

Since the roots of an equation are equal when and only when its discriminant equals zero (§ 98, II), the required values of k make $\Delta = 0$ and are the roots of
$$4\,k^2 - 4\,k - 8 = 0,$$
or
$$k^2 - k - 2 = 0.$$

Solve by factoring,
$$k^2 - k - 2 = (k-2)(k+1) = 0.$$

Thus the values of k are $\quad k = 2, \quad k = -1.$

Check: Substituting in the original equation for $k = 2$, we get
$$4\,x^2 + 4\,x + 1 = (2\,x + 1)^2,$$
and for $k = -1$ we get $\quad x^2 + 2\,x + 1 = (x+1)^2.$

(b) $x^2 + kx + 16 = 0$.

(c) $x^2 + 2x + k^2 = 0$.

(d) $x^2 - 2kx + 1 = 0$.

(e) $3kx^2 - 4x - 2 = 0$.

(f) $kx^2 - 3x + 4 = 0$.

(g) $x^2 + 4kx + k^2 + 1 = 0$.

(h) $k^2x^2 + 3x - 2 = 0$.

(i) $(k^2 + 3)x^2 + kx - 4 = 0$.

(j) $3kx^2 + kx - 1 = 0$.

(k) $x^2 + (3k + 1)x + 1 = 0$.

(l) $x^2 + 3x + k - 1 = 0$.

(m) $x^2 + 9kx + 6k + \frac{1}{4} = 0$.

(n) $4k^2x^2 + 4kx - 125 = 0$.

(o) $2x^2 - 4x - 2k + 3 = 0$.

(p) $(k + 1)x^2 + kx + k + 2 = 0$.

(q) $kx^2 + (4k + 1)x + 4k - 3 = 0$.

(r) $2(k + 1)x^2 + 3kx + k - 1 = 0$.

(s) $(k - 1)x^2 + 5kx + 6k + 4 = 0$.

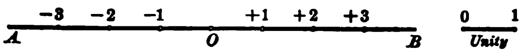
(t) $(2k + 3)x^2 - 7kx + \dfrac{6k - 7}{4} = 0$.

(u) $(k - 1)x^2 + (2k + 1)x + k + 3 = 0$.

CHAPTER IX

GRAPHICAL REPRESENTATION

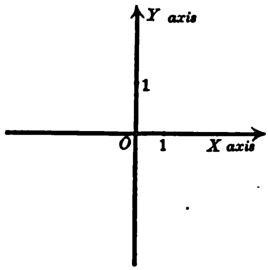
99. Representation of points on a line. Let us select on the indefinite straight line AB a certain point O as a point of reference. Let us also select a certain line, the length of which for the purpose in hand shall represent unity. Let us further agree that positive numbers shall be represented on AB by points to the right of O, whose distances from O are measured by the given



numbers, and negative numbers similarly by points to the left. Then there are certainly on AB points which represent such numbers as 2, -3, $\frac{1}{2}$, $-\frac{11}{16}$, or, in fact, any rational numbers. Since we can divide a line into any desired number of equal parts, we are able to find by geometrical construction the point corresponding to any rational number. Furthermore, by the principle that the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides, we can find the point corresponding to any irrational number expressed by square-root signs over rational numbers. More complicated irrational numbers cannot, however, in general be constructed by means of ruler and compasses, but *we assume that to every real number there corresponds a point on the line, and conversely, we assert that to every point on the line corresponds a real number.* This assumption of a **one-to-one correspondence** between points and real numbers is the basis of the graphical representation of algebraic equations.

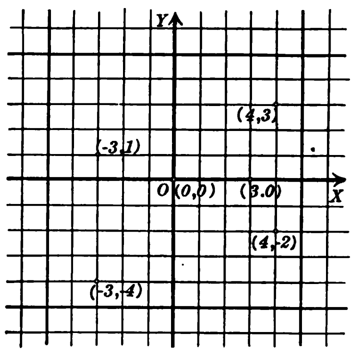
This amounts to nothing more than the assertion that every real number, rational or irrational, as, for instance, -6, $2+\sqrt{3}$, $\sqrt[9]{3}$, π, represents a certain distance from O on AB, and conversely, that whatever point on the line we may select, the distance from O to that point may be expressed by a real number.

100. Cartesian coördinates. We have seen that when the single letter x takes on real values all these values may be represented by points on a straight line.

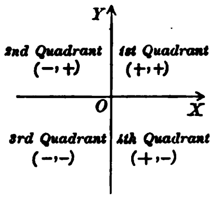


When, however, we have two variables, as x and y, which we wish to represent simultaneously, we make use of the plane. Just as we determined arbitrarily, on the line along which the single variable was represented, an arbitrary point for the point of reference and an arbitrary length for the unit distance, so now we select an indefinite line along which x shall be represented, and another perpendicular to it along which y shall be represented. The former we call the **X axis**; the latter the **Y axis.** The intersection O of these axes we take as the point of reference for each. This point is called the **origin.**

We select a unit of distance for x and a unit of distance for y which may or may not be the same, according to the problem under discussion. As before, we represent positive numbers on the X axis to the right, and negative numbers to the left. Positive values of y are represented above the X axis, and negative values below it. The arrowhead on the axes indicates the positive direction. Any pair of values of x and y, written (x, y), may now be represented by a point on the plane which is x units from the Y axis and y units from the X axis. Thus if $x = 0$, $y = 0$, written $(0, 0)$, the point represented is the origin. The point $(3, 0)$, i.e. $x = 3$, $y = 0$, is found by going three units of x to the right, i.e. in the positive direction of x and no units up. The point $(4, 3)$, i.e. $x = 4$, $y = 3$, is found by going four units of x to the right and three units of y up. The point $(-3, 1)$ is found by



going three units of x to the *left* and one unit of y *up*. The point $(-3, -4)$ is found by going three units of x to the *left* and four units of y *down*. In fact, if we let both x and y take on every possible pair of real values, we have a point of our plane corresponding to each pair of values of (x, y). Conversely, to every point of the plane correspond a pair of values of (x, y). These values are called the **coördinates** of the point. The value of x, i.e. the distance of the point from the Y axis, is called its **abscissa**; the value of y, i.e. the distance of the point from the X axis, is called its **ordinate**. If the point (x, y) is conceived as a moving point, and if no restriction is placed on the value of the coördinates so that they take on every possible pair of real values, every point in the plane is reached by the moving point (x, y).



The X and Y axes divide the plane into four parts called **quadrants**, which are numbered as in the figure. The proper signs of the coördinates of points in each of the quadrants are also indicated.

EXERCISES

The following exercises should be carefully worked on plotting paper, which can be bought ruled for the purpose.

1. Plot the points $(2, 3)$, $(0, 4)$, $(-4, 0)$, $(-9, -2)$, $(2, -4)$.

2. Plot with the aid of compasses the points $(1, \sqrt{2})$, $(\sqrt{3}, -\sqrt{2})$, $(2+\sqrt{3}, 2-\sqrt{3})$ $(-\sqrt{2}, -\sqrt{2})$.

3. Plot the square three of whose vertices are at $(-1, -1)$, $(-1, +1)$, $(+1, -1)$. What are the coördinates of the fourth vertex?

4. Plot the triangle whose vertices are $(2, 1)$, $(-6, -2)$, $(-4, 4)$.

5. Plot the two equilateral triangles two of whose vertices are $(6, 1)$, $(-6, 1)$. Find coördinates of the remaining vertices.

6. If the values of the coördinates (x, y) of a moving point are restricted so that both are positive and not equal to zero, where is the point still free to move?

7. If the coördinates (x, y) of a moving point are restricted so that continually $y = 0$, where is the point still free to move?

8. What is the abscissa of any point on the Y axis?

9. The coördinates of a variable point are restricted so that its ordinate is always 2. Where may the point move?

10. If both ordinate and abscissa of a point vanish, can the point move? Where will it be?

11. Plot the quadrilateral whose vertices are (0, 0), (– 6, – 3), (5, – 5), (– 1, – 8). What kind of a quadrilateral is it?

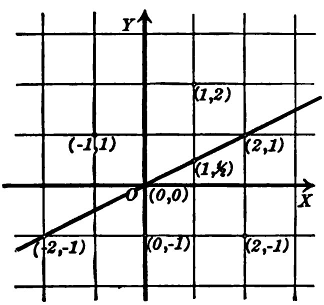
12. The coördinates of three vertices of a parallelogram are (– 1, – 1), (6, 2), (– 1, – 6). Find the coördinates of the fourth vertex.

13. The coördinates of two adjacent vertices of a square are (– 1, – 2) and (1, – 2). Find the coördinates of the remaining vertices (two solutions). Plot the figures.

14. The coördinates of two adjacent vertices of a rectangle are (– 1, – 2), (1, – 2). What restriction is imposed on the coördinates of remaining vertices?

15. The coördinates of the extremities of the bases of an isosceles triangle are (1, 6), (1, – 2). Where may the vertex lie? What restriction is imposed on the coördinates (x, y) of the vertex?

101. The graph of an equation. The equation $x = 2y$ is satisfied by numberless pairs of values (x, y); for example, (2, 1), (0, 0), (1, $\frac{1}{2}$), (– 2, – 1) all satisfy the equation. There are, however, numberless pairs of values which do not satisfy the equation; for example, (1, 2), (2, –1), (–1, 1), (0, – 1). The pairs of values which satisfy the equation may be taken as the coördinates of points in a plane. The totality of such points would thus in a sense *represent* the equation, for it would serve to distinguish the points whose coördinates do satisfy the equation from those whose coördinates do not.



After finding a few pairs of values which satisfy the above equation we note that any point whose abscissa is twice the ordinate, i.e. for which $x = 2y$, is a point whose coördinates satisfy the equation. Any such point lies on the straight line through the origin and the point (2, 1). We can then say that those points

and only those which are on the straight line represented in the figure have coördinates which satisfy the equation. This line is the **graphical representation** or **graph** of the equation.

The equation of a line or a curve is satisfied by the coördinates of every point on that line or curve.

Any point whose coördinates satisfy an equation is on the graph of the equation.

102. Restriction to coördinates. In § 100 it was seen that a moving point whose coördinates were unrestricted took on every position in the plane. We now see that when the coördinates of a point are restricted so as to satisfy a certain equation (as $x = 2y$), the motion of the point is no longer free, but restrained to move along a certain path. Thus, for instance, the equation $x = 4$ means that the path of the moving point is so restricted that its abscissa is always 4. Its ordinate is still unrestricted and may have any value. This shows that the plot of $x = 4$ is a straight line four units to the right of the Y axis and parallel to it, for the abscissa of every point on that line is 4, and every point whose abscissa is 4 lies on that line.

EXERCISES

Determine on what line the moving point is restricted to move by the following equations. Draw the graph.

1. $x = 6$.	**2.** $x = 0$.	**3.** $y = \frac{8}{7}$.
4. $x = y$.	**5.** $y = 2$.	**6.** $3x = y$.
7. $2x = y$.	**8.** $y = 0$.	**9.** $x = -\frac{3}{2}$.
10. $x = 3y$.	**11.** $3y = -x$.	**12.** $x + y = 0$.
13. $6x = 11$.	**14.** $y = -3$.	**15.** $2x = -3y$.
16. $x = -2y$.	**17.** $x = -1$.	**18.** $2x - 5y = 0$.

103. Plotting equations. Plotting an equation consists in finding the line or curve the coördinates of whose points satisfy the equation. Thus the process of § 101 was nothing else than plotting the equation $x = 2y$. This may be done in some cases by observing what restriction the equation imposes on the coördinates of the moving point; but more often we are obliged to form a

table of various solutions of the equation, and to form a curve by joining the points corresponding to these solutions. This gives us merely an approximate figure of the exact graph which becomes more accurate as we find the coördinates of points closer to each other on the line or curve.

RULE. *When y is alone on one side of the equation, set x equal to convenient integers and compute the corresponding values of y.*

Arrange the results in tabular form. Take corresponding values of x and y as coördinates and plot the various points.

Join adjacent points, making the entire plot a smooth curve.

When x is alone on one side of the equation integral values of y may be assumed and the corresponding values of x computed.

Care should be taken to join the points in the proper order so that the resulting curve pictures the variation of y when x increases continuously through the values assumed for it. By *adjacent* points we mean points corresponding to adjacent values of x.

Any scale of units along the X and Y axes that is convenient may be adopted. The scales should be so chosen that the portion of the curve that shows considerable curvature may be displayed in its relation to the axes and the origin.

When there is any question regarding the position of the curve between two integral values of x, an intermediate fractional value of x may be substituted, the corresponding value of y found, and thus an additional point obtained to fix the position of the curve in the vicinity in question.

EXERCISES

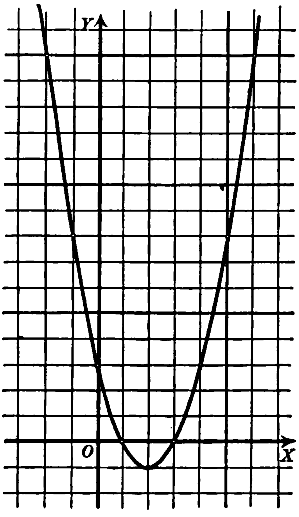
Plot:

1. $x^2 - 4x + 3 = y$.

Solution: In this equation if we set $x = 0$, 1, 2, 3, etc., we get 3, 0, -1, 0, etc., as corresponding values of y. Thus the points (0, 3), (1, 0), (2, -1), (3, 0), etc., are on the curve. These points are joined in order by a smooth curve.

x	y
0	3
1	0
2	-1
3	0
4	3
5	8
6	15

x	y
-1	8
-2	15



2. $y = x^2 - 7x + 1.$

3. $y = x^2 + 1.$

4. $y = x^2 - 3x + 2.$

5. $y = x^3 - 4x.$

6. $y = x^3 - 2x + 1.$

7. $y = x^2 + 6x + 5.$

8. $y = 2x^2 - 6x + 7.$

9. $y = 2x^2 - 3x + 4.$

10. $y = 2x^2 - 6x - 3.$

11. $y = x^2 - 12x + 11.$

104. Plotting equations after solution. When neither x nor y is already alone on one side of the equation, the equation should be solved for y (or x) and the rule of the previous section applied. It should be noted that when a root is extracted two values of y may correspond to a single value of x.

EXERCISES

Plot:

1. $2x^2 + 3y^2 = 9.$

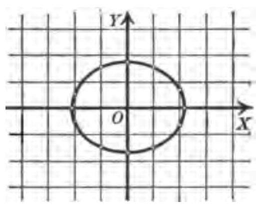
Solution

$$3y^2 = 9 - 2x^2,$$
$$y^2 = 3 - \tfrac{2}{3}x^2,$$
$$y = \pm\sqrt{3 - \tfrac{2}{3}x^2}.$$

Assuming the various integral values for x, we obtain the following table and plot:

x	y
0	$\pm\sqrt{3} = \pm 1.7$
1	$\pm\sqrt{\frac{7}{3}} = \pm 1.5$
2	$\pm\sqrt{\frac{1}{3}} = \pm .57$
$+3$	imaginary

x	y
-1	$\pm\sqrt{\frac{7}{3}} = \pm 1.5$
-2	$\pm\sqrt{\frac{1}{3}} = \pm .57$
-3	imaginary



In this example, when x is greater than 3 or less than -3, y is imaginary. Thus none of the curves is found outside a strip $x = \pm 3$.

To find exactly where the curve crosses the X axis, the equation may be solved for x and the value of x corresponding to $y = 0$ found. Thus

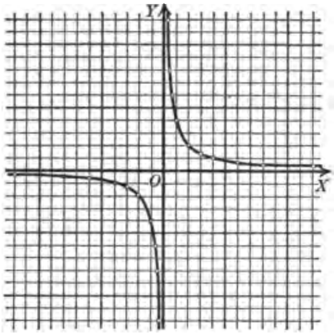
$$x = \pm\sqrt{\tfrac{9}{2} - \tfrac{3}{2}y^2}.$$

If $y = 0$, $x = \pm\sqrt{\frac{9}{2}} = 2.1$. This point is included in the plot.

2. $xy = 4.$

Solution:

$$y = \frac{4}{x}.$$



Form table for integral values of x.

x	y	x	y
1	4	-1	-4
2	2	-2	-2
3	$\frac{4}{3}$	-3	$-\frac{4}{3}$
4	1	-4	-1
6	$\frac{2}{3}$	-6	$-\frac{2}{3}$
8	$\frac{1}{2}$	-8	$-\frac{1}{2}$
12	$\frac{1}{3}$	-12	$-\frac{1}{3}$

Since this table does not give us any idea of the curves between $+1$ and -1, we supplement the table by assuming fractional values for x.

x	y	x	y
$\frac{2}{3}$	6	$-\frac{2}{3}$	-6
$\frac{1}{2}$	8	$-\frac{1}{2}$	-8
$\frac{1}{3}$	12	$-\frac{1}{3}$	$-\frac{1}{12}$

3. $x^2 = y^3$.

4. $xy = 16$.

5. $xy = -1$.

6. $xy = x + 1$.

7. $x^2 + y^2 = 16$.

8. $x^2 - y^2 = 9$.

9. $x^2 + y^2 = 25$.

10. $x^2 + x = 12y$.

11. $2xy + 3x = 2$.

12. $x^2 + 9y^2 = 36$.

13. $xy + y^2 = 10$.

HINT. $x = \dfrac{10 - y^2}{y}$.

14. $x - y + 2xy = 0$.

HINT. $x = \dfrac{y}{1 + 2y}$.

15. $6x^2 + 2x + 3y^2 = 0$.

16. $x^2 + 2x + 1 = y^2 - 3y$.

105. Graph of the linear equation. The intimate relation between the simplest equations and the simplest curves is given in the following theorems.

THEOREM I. *The graph of the equation $y = ax$ is the straight line through the origin and the point (1, a), where a is any real number.*

The proof falls into two parts.

First. Any point on the line through the origin and the point (1, a) has coördinates that satisfy the equation. Let P (Figure 1) with coördinates (x', y') be on the line OA. By similar triangles

$$\frac{a}{1} = \frac{y'}{x'},$$

or

$$y' = ax'.$$

Thus the coördinates of any point on the line satisfy the equation.

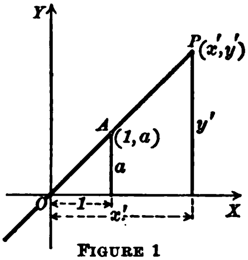


FIGURE 1

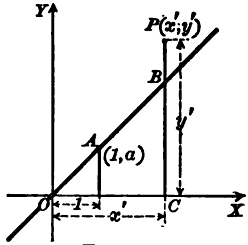


FIGURE 2

Second. Any point whose coördinates satisfy the equation lies on the line.

Let the coördinates (x', y') of the point P (Figure 2) satisfy the equation. Then we have

$$y' = ax',$$

or

$$\frac{y'}{x'} = a.$$

Let the ordinate y' cut the line at B. Then by the first part of the proof

$$BC = ax',$$

or

$$\frac{BC}{x'} = a.$$

Thus

$$a = \frac{y'}{x'} = \frac{BC}{x'}, \text{ or } y' = BC.$$

Hence P lies on the line.

THEOREM II. *The graph of any linear equation in two variables is a straight line.*

The general linear equation

$$Ax + By + C = 0 \tag{1}$$

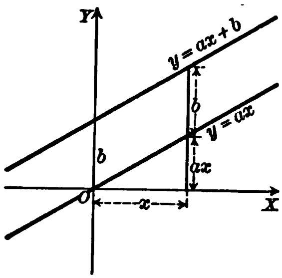
may be written in the form

$$y = ax + b, \tag{2}$$

where $a = -\frac{A}{B}$ and $b = -\frac{C}{B}$, provided $B \neq 0$ (§ 7). If $B = 0$, the equation $Ax + C = 0$ may be put in the form

$$x = -\frac{C}{A},$$

provided $A \neq 0$. This is evidently the equation of a straight line parallel to the Y axis (§ 102). If $B = 0$ and $A = 0$, we have no term left involving the variable. Thus the only case for which the theorem demands proof is when $B \neq 0$, and the equation may be reduced to form (2). By Theorem I we know that the graph of $y = ax$ is a straight line. If, then, we add to every ordinate y of the line $y = ax$ the constant b, the locus of the extremities of the lengthened ordinates will lie in a straight line, as one can easily prove by Geometry. But any point (x, y) on the upper line is such that its ordinate y is equal to the ordinate of the lower line, i.e. ax, and in addition the constant b; that is, $y = ax + b$. Also, since the upper line is the *locus* of the extremities of the lengthened ordinates, every point whose coördinates satisfy the equation $y = ax + b$ is on this upper line. Thus the equation (1) has a straight line as its graph.



Corollary. *Two lines whose equations are in the form*

$$y = ax + b, \tag{3}$$

$$y = ax + b' \tag{4}$$

are parallel.

For the value of the ordinates of (3) corresponding to a given abscissa, say x_1, is obtained from the ordinate of (4) corresponding to the same abscissa by adding the constant $b - b'$. Thus each point on (3) is always found $b - b'$ units above (below if $b - b'$ is negative) a point of (4). Thus the lines are parallel.

106. Method of plotting a line from its equation. Since the equation $y = ax + b$ is satisfied by the values $(0, b)$, the graph cuts the Y axis b units above (below if b is negative) the origin. Since it is satisfied by the values $(1, a + b)$, the graph passes through the point reached by going one unit of x to the right of $(0, b)$ and a units up (down if a is negative). These two points determine the line. We may then plot a linear equation by the following

RULE. *Reduce the given equation to the form*

$$y = ax + b.$$

Plot the point (0, b) as one of the two points that determine the line.

From this point go one unit of x to the right and a units of y up (down if a is negative) to find a second point that lies on the line.

Draw the line through these two points.

EXERCISES

Plot:

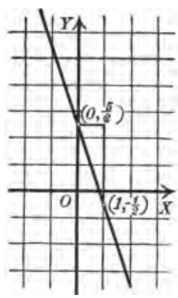
1. $6x + 2y - 5 = 0$.

Write in the form $y = ax + b$,

and we have $y = -3x + \frac{5}{2}$.

$a = -3$,

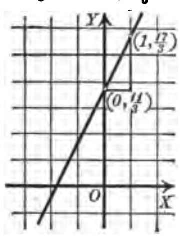
$b = \frac{5}{2}$.



Plot the point $(0, \frac{5}{2})$.

From this point go one unit of x to the right and three units of y down to find the second point, which helps determine the line.

2. $6x - 3y + 11 = 0$.
   $y = 2x + \frac{11}{3}$.



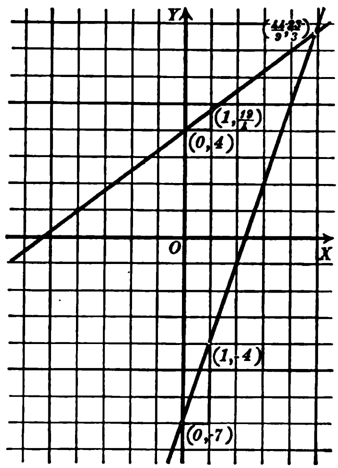
3. $x - y = 0$.
4. $x - y - 5 = 0$.
5. $x + y = 4$.
6. $2x = 6(1 - y)$.
7. $2x - y = 4$.
8. $12x + 10y = 5$.
9. $8x - y = 0$.
10. $15x - 10y = 4$.
11. $x - 8y = 16$.
12. $2x + y + 3 = 0$.
13. $x = 8(2 - y)$.
14. $2x - 6y - 1 = 0$.
15. $x - y - 1 = 0$.
16. $2x - 2y - 5 = 0$.
17. $x + y + 1 = 0$.
18. $3x - 6y - 4 = 0$.
19. $12x - 3y = 1$.
20. $7x - 8y - 9 = 0$.

107. Solution of linear equations, and the intersection of their graphs. The process of solving a pair of independent linear equations consists in finding a pair of numbers (x, y) which satisfy them both. Though each equation *alone* is satisfied by countless pairs of values (x, y), we have seen that there is only one pair that satisfy *both* equations. Since a pair of values which

satisfy an equation are the coördinates of a point on its graph, it appears that the pair of values that satisfy simultaneously two equations are the coördinates of the point common to the graphs of the two equations, that is, the coördinates of the points of intersection of the two lines.

EXERCISES

Find the solutions of the following equations algebraically. Verify the results by plotting and noting the coördinates of the point of intersection.



1. $3x - 4y + 16 = 0,$
$3x - y - 7 = 0.$

Solution:

$$\begin{array}{rr} 3x - 4y + 16 = 0 & (1) \\ \underline{3x - y - 7 = 0} & (2) \\ 3y - 23 = 0 & \\ y = \frac{23}{3}. & \end{array}$$

Substituting in (2),

$$3x = 7 + \tfrac{23}{3} = \tfrac{44}{3}.$$
$$x = \tfrac{44}{9}.$$

To plot (1) and (2) we get the equations in the form $y = ax + b$ and apply the rule. Thus

$$y = \tfrac{3}{4}x + 4.$$
$$y = 3x - 7.$$

2. $2x + 3y = 6,$ $7x + y = 2.$

3. $x + y = 5,$ $3x + y = 1.$

4. $x - 3y = -7,$ $4x + y = 11.$

5. $3x - 2y = 1,$ $3x + 2y = 5.$

6. $3x - 7y = 9,$ $x + 2y = 3.$

7. $x + y = -7,$ $2x - 3y = 6.$

8. $2x + y = 3,$ $3x - 7y = 1.$

9. $2x - 5y = 0,$ $x - y = 3.$

10. $x + 2y = -10,$ $2x - y = 0.$

11. $x + y = -4,$ $4x - 3y = 5.$

12. $x - y = 7,$ $2x - 3y = 3.$

13. $x - y = 1,$ $2x - 4y = -16.$

14. $x + y = 5,$ $4x - 2y = 28.$

15. $x - y = 2,$ $4x - 5y = 9.$

16. $6x - 5y = 5,$ $2x + 3y = -20.$

17. $x - y = -4,$ $2x + 6y = 16.$

18. $3x + 2y = 9,$ $3x - y = 2.$

19. $2x + 6y = -20,$ $3x + y = 2.$

108. Graphs of dependent equations. We have defined (§ 57) dependent equations as those that are reducible to the same form on multiplying or dividing by a constant. Thus two dependent equations are reducible to the same equation of the form $y = ax + b$. Hence ***dependent equations have as their graphs the same straight line.*** We see now the geometrical meaning of the statement that dependent equations have countless common solutions. Since their graphs have not one but countless points in common, being the same line, it is clear that the coördinates of these countless points will satisfy both equations.

109. Incompatible equations. By our definition (§ 60) incompatible equations have no common solution. Since every pair of distinct lines have a common point unless they are parallel, we can foresee the

THEOREM. *Incompatible equations have parallel lines as graphs.*

Let the equations

$$ax + by + c = 0, \qquad (1)$$

$$a'x + b'y + c' = 0 \qquad (2)$$

be incompatible. This is true (§ 60) when and only when

$$ab' - a'b = 0. \qquad (3)$$

Let us then assume (3).

Write (1) and (2) in the form

$$y = -\frac{a}{b}x - \frac{c}{b}. \qquad (4)$$

$$y = -\frac{a'}{b'}x - \frac{c'}{b'}. \qquad (5)$$

This may be done if neither b nor b' equals zero. If both b and b' vanish, the lines (1) and (2) are both parallel to the Y axis and hence to each other, which was to be proved. But if only one of them vanishes, say $b = 0$, then by (3) $a = 0$ (§ 5), in which case (1) does not include either variable. Thus we may assume that neither b nor b' vanishes and that (4) and (5) may be obtained from (1) and (2).

By (3) $$\frac{a}{b} = \frac{a'}{b'}.$$

Our equations (4) and (5) become

$$y = -\frac{a}{b}x - \frac{c}{b},$$

$$y = -\frac{a}{b}x - \frac{c'}{b'},$$

which represent parallel lines, by the Corollary, p. 96.

This theorem completes the discussion of the graphical representation of the possible classes (§ 61) of pairs of linear equations.

EXERCISES

Plot and solve:

1. $8x + 2y = 3,$ $4x + y = 8.$

2. $2x + 6y = 1,$ $x + 3y = 7.$

3. $10x - 5y = 15,$ $2x - y = 3.$

4. $2x - 3y = 6,$ $8x - 12y = 24.$

5. $x - 7y = 1,$ $4x - 28y = 56.$

6. $12x - 6y = 18,$ $2x - y = 1.$

7. $x - 3y = 2,$ $6x - 18y = 36.$

8. $2x - 3 + y = 0,$ $4x - 7 + 2y = 0.$

110. Graph of the quadratic equation. Let

$$y = ax^2 + bx + c, \tag{1}$$

where as usual a, b, and c represent integers and a is positive.

If we let x take on various values, y will have corresponding values and we may plot the equation as in § 103. A root of the quadratic equation

$$ax^2 + bx + c = 0 \tag{2}$$

is a number which substituted for x satisfies the equation, that is, gives the value $y = 0$ in (1). Thus the points on the graph of (1) which represent the roots of the equation (2) are the points for which $y = 0$, that is, where the curve crosses the X axis. The numerical value of the roots is the measure of the distance along the X axis from the origin to the points where the curve cuts the axis. Since this distance is always a real distance, only real roots are represented in this manner.

THEOREM. *If the graph of* (1) *has no point in common with the X axis, the equation* (2) *has imaginary roots, and conversely.*

Every equation of form (2) has two roots either real or imaginary (§ 89). If the graph of (1) has no point in common with the X axis, there is no *real* value of x for which $y = 0$, i.e. no real root of (2). The roots must then be imaginary.

Conversely, if (2) has only imaginary roots, there is no real value of x which satisfies it, i.e. which makes $y = 0$ in (1). Thus the curve has no point in common with the X axis.

This suggests the following universal

PRINCIPLE. *Non-intersection of graphs corresponds to imaginary or infinite-valued solutions of equations.*

111. Form of the graph of a quadratic equation. Consider the equation

$$y = 2x^2 + 7x + 2. \qquad (1)$$

By substituting for x a very large positive or negative number, say $x = \pm 100$, y is large positively. Thus for values of x far to the right or left the curve lies far above the X axis. If we assign to y a certain value, say $y = 2$, we can find the corresponding values of x by solving a quadratic equation. Thus in (1) let $y = 2$.

$$2 = 2x^2 + 7x + 2,$$

or

$$2x^2 + 7x = 0.$$

The roots are $\quad x_1 = -3\tfrac{1}{2}, \quad x_2 = 0.$

Hence the points $(-3\tfrac{1}{2}, 2)$ and $(0, 2)$ are on the curve (§ 101). That is, if we go up two units on the Y axis, the curve is to be found three and one half units to the left and also again on the Y axis. If in (1) we let $y = -4$, the corresponding values of x are very nearly equal to each other ($-1\tfrac{1}{2}$ and -2), which means that the curve meets a line parallel to the X axis and four units below it at points very near together. The question now arises, Where is the bottom of the loop of the curve? This lowest point of the loop has as its value of y that number to which correspond *equal* values of x. Hence we must determine for what value of y the equation (1), that is, the equation

$$2x^2 + 7x + (2 - y) = 0,$$

has equal roots. Comparing with the equation $ax^2 + bx + c = 0$ (§ 98), we have

$$2 = a, \ 7 = b, \ 2 - y = c.$$

Thus the condition $b^2 - 4ac = 0$ becomes

$$49 - 4 \cdot 2\,(2 - y) = 0,$$

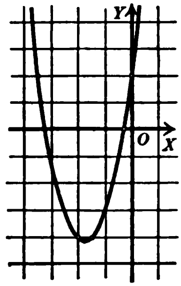
or

$$y = -\frac{49 - 16}{8} = -\frac{33}{8} = 4\tfrac{1}{8}.$$

Substituting this value of y in (1), we get $-\frac{7}{4}$ as the corresponding value of x.

This gives a single value of y for which the values of x are equal; hence the graph of (1) is a single festoon as in the figure.

y	x
0	$-.3+$ or $-3.2+$
2	0 or $-3\frac{1}{2}$
-4	$-1\frac{1}{2}$ or -2
$-4\frac{1}{8}$	$-1\frac{3}{4}$



If we take the general equation

$$ax^2 + bx + c = y,$$

we find precisely similarly that the bottom of the loop is at a point whose ordinate is

$$y = -\frac{b^2 - 4ac}{4a} = -\frac{\Delta}{4a}.$$

Thus we see again that if the discriminant is negative the graph is entirely above the X axis and both roots are imaginary (§§ 98, 110), since the ordinate of the lowest point of the loop is positive. If the discriminant is positive, the graph cuts the X axis and both roots are real.

The results of this section enable us to determine a value of y from the coefficients which determine the lowest point of the loop of the curve precisely, and hence to show beyond question from the graph whether the equation has real or imaginary roots.

EXERCISES

Plot the following equations and determine by measurement the roots in case they are real. Find in each case the lowest point on the loop.

1. $x^2 + x + 1 = y$.
2. $x^2 - 4x + 7 = y$.
3. $x^2 - 6x + 10 = y$.
4. $x^2 + 7x + 6 = y$.
5. $x^2 - 6x + 9 = y$.
6. $3x^2 - 7x - 6 = y$.
7. $x^2 - 6x + 1 = y$.
8. $x^2 - 6x + 5 = y$.
9. $2x^2 - 9x + 7 = y$.
10. $x^2 + 2x - 1 = y$.
11. $x^2 - 4x + 4 = y$.
12. $3x^2 - 4x - 3 = y$.
13. $2x^2 - x - 3 = y$.
14. $3x^2 + 8x + 5 = y$.
15. $4x^2 + 12x + 9 = y$.
16. What is the characteristic feature of the plot of an equation whose roots are equal?

112. The special quadratic $ax^2 + bx = 0$. When in the quadratic equation

$$ax^2 + bx + c = 0, \tag{1}$$

$c = 0$, we can always factor the equation into

$$ax^2 + bx = x(ax + b) = 0,$$

or

$$x\left(x + \frac{b}{a}\right) = 0.$$

Thus the roots are $\qquad x_1 = 0,\ x_2 = -\frac{b}{a}.$

Conversely, if $x = 0$ is a root, then (§ 95) $x - 0$, or x, is a factor and the equation can have no constant term.

This affords the

THEOREM. *A quadratic equation has a root equal to zero when and only when the constant term vanishes.*

We show in a similar manner that both roots of the equation (1) are zero when and only when $b = c = 0$.

EXERCISES

1. Prove the theorem just given by considering the expressions for the roots in terms of the coefficients (§ 89).

2. For what real values of k do the following equations have one root equal to zero?

(a) $x^2 + 6x - k + 1 = 0$. (b) $2x^2 - 3x + k^2 - 1 = 0$.

(c) $x^2 + 6x + k^2 + 1 = 0$. (d) $2x^2 - 4x + k^2 - 3k = 0$.

(e) $2x^2 + 2kx - 2k^2 - 4k - 2 = 0$. (f) $6x^2 - 4x + 2k^2 + k + 7 = 0$.

3. What is the characteristic feature of the plot of an equation which has one root equal to zero?

4. For what real value of k will both roots of the following equations vanish?

(a) $\dfrac{x^2}{k} + 3x - 1 = 0$. (b) $x^2 + (k^2 + 3)x + k - 3 = 0$.

(c) $x^2 + (k^2 + 1)x + 1 = 0$. (d) $x^2 + (k - 3)x + 2k^2 - 5k - 3 = 0$.

(e) $x^2 + (k + 1)x + k^2 - 1 = 0$. (f) $(k-3)x^2 + (k^2-9)x + k^2 - 4k + 3 = 0$.

113. The special quadratic $ax^2 + c = 0$. This equation may be written in the form $x^2 + \frac{c}{a} = 0$ and factored * immediately into

$$\left(x + \sqrt{-\frac{c}{a}}\right)\left(x - \sqrt{-\frac{c}{a}}\right) = 0,$$

which shows that the roots are equal numerically but have opposite signs. The roots are

$$x_1 = \sqrt{-\frac{c}{a}}, \quad x_2 = -\sqrt{-\frac{c}{a}}.$$

Since in the equation $ax^2 + c = y$ the variable x occurs only in the term x^2, we get the same value of y for positive and negative values of x. Hence the loop which forms the graph of the equation is symmetrical with respect to the Y axis.

114. Degeneration of the quadratic equation. The equation

$$ax^2 + bx + c = 0$$

has the roots

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We wish to find the effect on the roots x_1 and x_2 when a becomes very small. If we let a approach 0, then x_1 approaches an expression of the form $\frac{0}{0}$, which must always be avoided. Rationalize the numerators and we get

$$x_1 = \frac{4ac}{2a\left(-b - \sqrt{b^2 - 4ac}\right)} = \frac{2c}{-b - \sqrt{b^2 - 4ac}},$$

$$x_2 = \frac{4ac}{2a\left(-b + \sqrt{b^2 - 4ac}\right)} = \frac{2c}{-b + \sqrt{b^2 - 4ac}}.$$

As a approaches 0, evidently $b^2 - 4ac$ approaches b^2, x_1 approaches $-\frac{c}{b}$, and x_2, since its denominator becomes very small,

* When $-\frac{c}{a}$ is positive this involves real factors. If $-\frac{c}{a}$ is negative the factors are imaginary (§ 152).

increases without limit, that is, approaches infinity. Thus the quadratic equation approaches a linear equation when a approaches 0, and one of its roots disappears since it has increased in value beyond any finite limit. The loop-shaped graph of the quadratic equation must then approach a straight line as a limit when a approaches 0. This is made clear from the following figure, where a has the successive values 1, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{50}$, 0.

In the figure the curves represent the following equations:

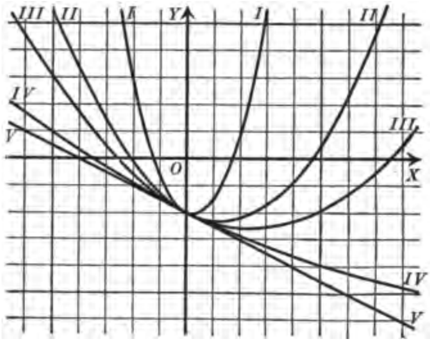
$$x^2 - \frac{x}{2} - 2 = y. \qquad \text{(I)}$$

$$\frac{x^2}{5} - \frac{x}{2} - 2 = y. \qquad \text{(II)}$$

$$\frac{x^2}{10} - \frac{x}{2} - 2 = y. \qquad \text{(III)}$$

$$\frac{x^2}{50} - \frac{x}{2} - 2 = y. \qquad \text{(IV)}$$

$$-\frac{x}{2} - 2 = y. \qquad \text{(V)}$$



In a similar manner we can show that when in the equation $bx + c = 0$, b approaches 0 as a limit, the root of the linear equation becomes infinite.

EXERCISES

1. What real values must k approach as a limit in order that one root of each of the following equations may become infinite?

(a) $kx^2 + 6x + 1 = 0.$ (b) $(k^2 + 1)x^2 + x + 1 = 0.$

(c) $(kx - 1)^2 - (x + 2)^2 = (k + x)^2.$ (d) $k^2 + 4k^2x^2 - (x - 1)^2 + 2 = 0.$

(e) $\sqrt{2kx - 1} + \sqrt{6k - 1} = \sqrt{kx + 1}.$ (f) $\sqrt{x - k} + \sqrt{x + k} = \sqrt{kx + 1}.$

(g) $\frac{x - 1}{k - 1} + \frac{k + 1}{x + 1} + \frac{x}{k^2} = 0.$ (h) $\sqrt{\frac{kx - 1}{kx + 1}} + \sqrt{\frac{kx + 1}{kx - 1}} = \sqrt{\frac{4k - 1}{k^2}}.$

(i) $\frac{(x - 1)^2}{(x + 1)^2} + \frac{(k - 1)}{(k + 1)} - \frac{1}{k} = 0.$

(j) $(k^2 - 1)x^2 + (k - 1)x + k^2 + 4k - 5 = 0.$

2. What real values must k and m approach as a limit in order that both roots of the following may become infinite?

(a) $kx^2 + mx + 1 = 0.$

(b) $(2k - m)x^2 + kx - 2 = 0.$

(c) $2kx^2 + (3m - 1 + k)x = 8x^2 - 1.$

(d) $(k - 1)x^2 + (k + m + 1)x + 3 = 0.$

(e) $x^2 - x - 2(k + m)x = (k + m)(x^2 - 1).$

(f) $(k + m)x^2 + 2(k + m) + 1 = x^2 - 2x.$

(g) $(k + m + 1)x^2 + (2k - m - 1)x + 1 = 0.$

(h) $(2k + m + 2)x^2 + (4k + 2m + 3)x + 3 = 0.$

115. Sum and difference of roots. Let x_1 and x_2 be the roots of

$$x^2 + bx + c = 0. \tag{1}$$

Then (§ 95) $x - x_1$ and $x - x_2$ are factors, and their product $x^2 - (x_1 + x_2)x + x_1x_2$ is exactly the left-hand member of (1). Consequently the equation

$$x^2 + bx + c = x^2 - (x_1 + x_2)x + x_1x_2$$

is true for all values of x. Hence by § 96

$$-(x_1 + x_2) = b, \tag{2}$$

$$x_1x_2 = c. \tag{3}$$

We may state these facts in the

THEOREM. *The coefficient of x in the equation $x^2 + bx + c = 0$* * *is equal to the sum of its roots with their signs changed.*

The constant term is equal to the product of the roots.

EXERCISES

1. Prove the statement just made from the expression for the roots in terms of the coefficients (§ 89).

2. Form the equations whose roots are the following:

(a) $6, 1.$ (b) $\frac{1}{2}, \frac{1}{2}.$ (c) $\frac{2}{5}, 3.$ (d) $-\frac{1}{6}, -6.$

(e) $\frac{1}{2}, \frac{1}{3}.$ (f) $-\frac{2}{3}, +\frac{1}{2}.$ (g) $2 + \sqrt{3}, 2 - \sqrt{3}.$ (h) $-\sqrt{3}, \sqrt{3}.$

* We should for the present exclude the case where $b^2 - 4c < 0$, since the roots x_1 and x_2 are then imaginary and we have not as yet defined what we mean by the sum or the product of imaginary numbers. We shall see later that the theorem is also true in this case.

3. If 4 is one root of $x^2 - 3x + c = 0$, what value must c have?

Solution: Let x_1 be the remaining root.

Then by (1) $$-(x_1 + 4) = -3,$$

or $$x_1 = -1.$$

By (2) $$c = x_1 \cdot 4 = (-1)4 = -4.$$

4. Find the value of the literal coefficients in the following equations.

(a) $x^2 + bx - 9 = 0$. One root is 3.

(b) $x^2 + 4x + c = 0$. One root is 2.

(c) $ax^2 + 3x - 4 = 0$. One root is 2.

(d) $ax^2 + 3x + 4 = 0$. One root is 7.

(e) $ax^2 + 2x + 6 = 0$. One root is 6.

(f) $x^2 + bx + 4 = 0$. One root is -1.

(g) $x^2 - bx - 6 = 0$. One root is -3.

(h) $x^2 + bx + 6 = 0$. One root is -6.

(i) $2x^2 - 6x - c = 0$. One root is -4.

(j) $x^2 - 6x + c = 0$. One root is double the other.

(k) $x^2 + c = 0$. The difference between the roots is 8.

(l) $x^2 - 5x + c = 0$. One root exceeds the other by 3.

(m) $x^2 - 7x + c = 0$. The difference between the roots is 6.

(n) $x^2 - 6x + c = 0$. The difference between the roots is 4.

(o) $x^2 - 3x + c = 0$. The difference between the roots is 2.

(p) $x^2 - 2x + c = 0$. The difference between the roots is 8.

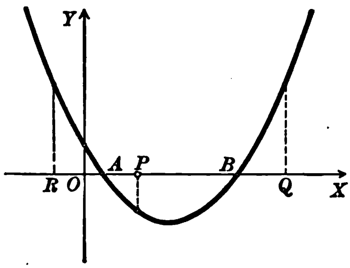
116. Variation in sign of a quadratic. It is often necessary to know the sign of the expression
$$ax^2 + bx + c$$
for certain real values of x, and to determine the limits between which x may vary while the expression preserves the same sign. We assume as usual that a is positive.

THEOREM I.* *If the discriminant of $ax^2 + bx + c$ is positive, the quadratic is negative for all values of x between the values of the roots of the equation. For other values of x (excepting the roots) the quadratic is positive.*

* If a were negative, Theorem I would read as follows: *If the discriminant is positive, the quadratic is positive for all values of x between the values of the roots of the equation. For other values of x (excepting the roots) the quadratic is negative.*

When a is negative Theorems II and III may be modified in an analogous manner.

In § 98 we found that when the discriminant of a quadratic equation is positive the equation has two real roots. If two roots are real, the loop of the graph of the equation $ax^2 + bx + c = y$ cuts the X axis in two points (§ 110) as in the figure. The roots are represented by A and B, and any real value of x between the roots is represented by a point P in the line AB. Since the curve is below the X axis at any such point, the value of y, i.e. of the expression $ax^2 + bx + c$ for values of x between the roots, is negative.



The value of the expression for any value of x greater or less than both roots is seen to be positive, since for such points, for example Q and R, the graph is above the X axis.

THEOREM II. *If the discriminant of $ax^2 + bx + c$ is negative, the expression is positive for all real values of x.*

When the discriminant is negative the entire graph of $ax^2 + bx + c = y$ is above the X axis (§ 111), and consequently for any real value of x the corresponding value of y, i.e. the value of $ax^2 + bx + c$, is positive.

THEOREM III. *If the discriminant of $ax^2 + bx + c$ is zero, the value of the expression is positive for all values of x except the roots of the equation $ax^2 + bx + c = 0$.*

Hint. See example 16, p. 102.

We may restate these three theorems and prove them algebraically as follows:

THEOREM IV. *If the discriminant of the quadratic $ax^2 + bx + c$ is positive, the values of the quadratic and a differ in sign for all values of x lying between the roots, and agree for other values.*

If the discriminant is zero or negative, the value of the quadratic always agrees with a in sign.

CASE I. Since the discriminant is positive, the equation $ax^2+bx+c=0$ has two unequal real roots, as x_1 and x_2, of which we will assume x_1 is the greater, and we may write the quadratic in the form

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Now for any value of x between x_1 and x_2 the factor $x - x_1$ is negative, while $x - x_2$ is positive, which shows that the quadratic is opposite in sign to a for such values of x. For other values of x both these factors are either positive or negative, and for such values the quadratic is of the same sign as a.

CASE II. Since the discriminant $b^2 - 4ac$ is negative and the roots are of the form $-\dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$, we may write the quadratic

$$ax^2 + bx + c = a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]^*$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right].$$

Now for any value of x the expression $\left(x + \dfrac{b}{2a}\right)^2$ is positive, and since $b^2 - 4ac$ is negative, $4ac - b^2$ is positive; and we observe that the last member of the equation has the same sign as a.

CASE III. Since the discriminant is zero, the roots are equal and the expression has the form

$$ax^2 + bx + c = a(x - x_1)^2,$$

which has evidently the same sign as a, for any value of x.

EXERCISES

1. Between what values of x is the expression $\sqrt{x^2 - 5x + 4}$ imaginary?

Solution: The roots of $x^2 - 5x + 4 = 0$ are 4 and 1.

The discriminant $\Delta = b^2 - 4ac = 25 - 16 = 9$ is positive.

Thus by Theorem I or IV, if $1 < x < 4$† the expression under the radical sign is negative and the whole expression is imaginary.

2. For what values of k are the roots of

$$(k + 3)x^2 + kx + 1 = 0 \qquad (1)$$

(a) real and unequal? (b) imaginary?

Solution: $a = k + 3,\ b = k,\ c = 1.$

$$\Delta = b^2 - 4ac = k^2 - 4(k + 3) = k^2 - 4k - 12.$$

(a) If $\Delta > 0$, the roots of (1) are real and unequal.
The roots of $k^2 - 4k - 12$ are $k = -2$ and 6
Then, by Theorem II, if

$$k < -2 \text{ or } k > 6,\ \Delta > 0.$$

(b) By Theorem I, if $\quad -2 < k < 6,\ \Delta < 0,$

and the roots of (1) are imaginary.

* See § 152. † Read "1 is less than x which is less than 4" or "x is between 1 and 4."

3. Determine for what values of x the following expressions are negative.

(a) $x^2 + 2x - 1$.
(b) $x^2 - 6x + 4$.
(c) $x^2 - 11x + 10$.
(d) $x^2 - 17x + 60$.
(e) $-x^2 - 2x + 1$.
(f) $-x^2 + 7x + 30$.

4. Determine for what values of k the roots of the following equations are (a) real and unequal, (b) imaginary.

(a) $3kx^2 - 4x - 2 = 0$.
(b) $x^2 + 4kx + k^2 + 1 = 0$.
(c) $x^2 + 9kx + 6k + \frac{1}{4} = 0$.
(d) $x^2 + (3k + 1)x + 1 = 0$.
(e) $(k^2 + 3)x^2 + kx - 4 = 0$.
(f) $2x^2 - 4x - 2k + 3 = 0$.
(g) $kx^2 + (4k + 1)x + 4k - 3 = 0$.
(h) $(k - 1)x^2 + 5kx + 6k + 4 = 0$.
(i) $(k - 1)x^2 + (2k + 1)x + k + 3 = 0$.

CHAPTER X

SIMULTANEOUS QUADRATIC EQUATIONS IN TWO VARIABLES

117. Solution of simultaneous quadratics. A single equation in two variables, as $x^2 + y^2 = 5$, is satisfied by many pairs of values, as (1, 2), $(\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}})$, (2, 1), and so on, though there are at the same time numberless pairs of values that do not satisfy it, as (0, 1), (1, 1), (2, 3). Thus the condition that (x, y) satisfy a single quadratic equation imposes a considerable restriction on the values that x and y may assume. If we further restrict the value of the pair of numbers (x, y) so that they also satisfy a second equation, the number of solutions is still further limited. The problem of solving two simultaneous equations consists in finding the pairs of numbers that satisfy them both.

118. Solution by substitution. In this method of solution the restriction imposed on (x, y) by one equation is imposed on the variables in the other equation by substitution.

EXAMPLE. Solve
$$2x^2 + y^2 = 1, \quad (1)$$
$$x - y = 1. \quad (2)$$

Solution: Equation (2) states that $x = 1 + y$. Thus our desired solution is such a pair of numbers that (1) is satisfied and at the same time x is equal to $y + 1$.

If we substitute in (1) $1 + y$ for x, we are imposing on its solution the restriction implied by (2).

Thus
$$2(1 + y)^2 + y^2 = 1,$$
or
$$3y^2 + 4y + 1 = 0.$$
The roots are
$$y = -1, \quad y = -\tfrac{1}{3}.$$

Corresponding to $y = -1$ we get from (2) $x = 0$.
Corresponding to $y = -\frac{1}{3}$ we get from (2) $x = \frac{2}{3}$.
Thus the solutions are $(0, -1)$ and $(\frac{2}{3}, -\frac{1}{3})$.

EXERCISES

Solve the following:

1. $x + y = 5,\quad xy = 4.$

2. $x - y = 5,\quad xy = 36.$

3. $x + y = a,\quad x^2 + y^2 = bxy.$

4. $3x - y = 5,\quad xy - x = 0.$

5. $x + y = a,\quad x^2 + y^2 = b.$

6. $x^2 + y^2 = 50,\quad 9x + 7y = 70.$

7. $x^2 + y^2 = 40,\quad x - 3y = 0.$

8. $2x - 3y = 4,\quad x^2 - y^2 = 0.$

9. $xy = 12,\quad 2x + 3y = 18.$

10. $x : y = 9 : 4,\quad x : 12 = 12 : y.$

11. $x^2 : y^2 = a^2 : b^2,\quad a - x = b - y.$

12. $5x^2 + y = 3xy,\quad 2x - y = 0.$

13. $x^2 - xy + y^2 = 7,\quad 2x - 3y = 0.$

14. $(x + y)(x - 2y) = 7,\quad x - y = 3.$

15. $3x^2 - 4y = 5x - 2y^2,\quad 3x + 4y = 10.$

16. $x^2 + y = y^2 + x - 18,\quad x : y = 2 : 3.$

17. $ax - by = cy,\quad a^2x^2 - b^2y^2 = acxy + m^2.$

18. $x^2 + 2xy + y^2 = 7(x - y),\quad 2x - y = 5.$

19. $ax^2 + (a - b)xy - by^2 = c^2,\quad (x + y) : (x - y) = a : b.$

20. $2x^2 - 5xy + y^2 + 10x + 12y = 100,\quad 2x - 3y = 1.$

21. $7(x + 5)^2 - 9(y + 4)^2 = 118,\quad x - y = 1.$

22. $x^2 + y^2 = a^2,\quad \dfrac{x}{y} = \dfrac{m}{n}.$

23. $x^2 + y^2 = 130,\quad \dfrac{x + y}{x - y} = 8.$

24. $\dfrac{x^2 + y + 1}{y^2 + x + 1} = \dfrac{3}{2}.\quad x - y = 1.$

25. $\dfrac{2x - y + 1}{x - 2y + 1} = \dfrac{8}{3},\quad x^2 - 3xy + y^2 = 5.$

26. $\dfrac{1 + x + x^2}{1 + y + y^2} = 3\quad x + y = 6.$

27. $xy + 72 = 6(2x + y),\quad \dfrac{x}{y} = \dfrac{2}{3}.$

28. $\dfrac{3x - 2}{y + 5} + \dfrac{y}{x} = 2.\quad x - y = 4.$

29. $\dfrac{4x+y-1}{2x+y-1} - \dfrac{4x+y-12}{2x+y-12} = 2\tfrac{2}{3},\quad 3x + y = 13.$

30. $x(x - y) - 5y = 6\quad \dfrac{x}{x - y} = 3\tfrac{1}{2}.$

31. $\dfrac{10}{x + 2} + \dfrac{9}{y - 1} = 5,\quad \dfrac{2}{x - 1} = \dfrac{4}{y}.$

32. $\dfrac{8}{x} + \dfrac{3}{y} = 3,\quad 5(y - 1) = 2(x + 1).$

119. Number of solutions. We have proved (p. 42) that two linear equations have in general one and only one solution.

THEOREM. *A quadratic equation and a linear equation have in general two and only two solutions.*

If the linear equation is solved for one variable, say x, and this is substituted in the quadratic equation, we get a quadratic equation to determine all possible values of the other variable (i.e. y), which must in general be two in number (§ 98). To each one of these values of y will correspond one and only one value of x, thus affording two solutions of the pair of equations.

EXERCISES

1. When may, as a special case, a quadratic and a linear equation have only one solution?

2. When may a quadratic and a linear equation have imaginary solutions?

3. Find the real values of k for which the following equations have (1) only one solution, (2) imaginary solutions.

(a) $$x^2 + y^2 = 16, \quad (1)$$
$$x - y = k. \quad (2)$$

Solution: $$x = y + k.$$

Substitute in (1), $$(y + k)^2 + y^2 = 16.$$

or $$2y^2 + 2ky + k^2 - 16 = 0.$$

As in § 98, $$a = 2;\ b = 2k;\ c = k^2 - 16.$$

Hence $$\Delta = b^2 - 4ac = 4k^2 - 8k^2 + 128 = -4k^2 + 128.$$

(1) $\Delta = 0$ when $4k^2 = 128$,

or $k = \pm 4\sqrt{2}$. There is then only one solution.

(2) $\Delta < 0$ when $k^2 > 32$. The solution is then imaginary.

(b) $xy = 1,\ x + y = k.$ (c) $x + ky = 5,\ x^2 + y^2 = 5.$ (d) $x^2 = 8y,\ 2x - y = k.$

(e) $y^2 = kx,\ x - 5y = 1.$ (f) $y - x = k,\ x^2 - 4y = 0.$ (g) $x^2 - y^2 = 9,\ x - 2y = k.$

(h) $x^2 - y^2 = 8,\ x + y = k.$ (i) $x^2 + y^2 = 25,\ 4x - 3y = k.$ (j) $2x^2 + 3y^2 = 6,\ x - ky = 1.$

120. Solution when neither equation is linear. In the examples previously given one equation has been linear and the other quadratic in one or both variables. Often when neither of the original equations is linear a pair of equivalent (p. 41) equations one or both of which are linear may be found. These latter equations may be solved by substitution.

EXERCISES

Solve the following equations.

When neither equation is linear, we can often obtain by addition an equation from which by the extraction of the square root a linear equation may be found.

1. $$x^2 + y^2 = 17, \quad (1)$$
$$xy = 4. \quad (2)$$

Solution: $x^2 + y^2 = 17$ (3)

Multiply (2) by 2, $2xy = 8$ (4)

Add $x^2 + 2xy + y^2 = 25$

Extract the square root, $x + y = \pm 5.$ (5)

Subtract (4) from (3), $x^2 - 2xy + y^2 = 9.$

Extract the square root, $x - y = \pm 3.$ (6)

Solve (5) and (6) as simultaneous equations,

$$x + y = \pm 5,$$
$$x - y = \pm 3.$$
$$x = +4, \; +1, \; -1, \; -4.$$
$$y = +1, \; +4, \; -4, \; -1.$$

Thus the solutions are four in number, $(4, 1), (1, 4), (-1, -4), (-4, -1)$.

The following exercise affords another case where a linear equation may be found by addition and extraction of the square root.

2. $$x^2 + xy = 6, \quad (1)$$
$$xy + y^2 = 10. \quad (2)$$

Solution: Add (1) and (2),

$$x^2 + 2xy + y^2 = 16.$$

Extract the square root, $x + y = \pm 4.$

Substitute in (1), $x^2 + x(\pm 4 - x) = 6,$

$$x^2 \pm 4x - x^2 = 6,$$
$$x = \pm \tfrac{3}{2} = \pm 1\tfrac{1}{2}.$$

Substitute in (4), $y = 2\tfrac{1}{2}, \; -2\tfrac{1}{2}.$

Thus our solutions are $(-\tfrac{3}{2}, -2\tfrac{1}{2}), (\tfrac{3}{2}, +2\tfrac{1}{2})$.

When neither of the original equations is quadratic, we can often find by division an equivalent pair of equations one of which is linear and the other quadratic, as in the following exercise.

3.
$$x^3 + y^3 = 12, \tag{1}$$
$$x + y = 2. \tag{2}$$

Solution: Divide (1) by (2),
$$x^2 - xy + y^2 = 6. \tag{3}$$

Square (2), $\quad x^2 + 2xy + y^2 = 4$

Subtract, $\quad -3xy = 2$
$$xy = -\tfrac{2}{3}. \tag{4}$$

Solve (4) with (2) by substitution.

When the sum of the exponents of the variables is the same in every term, the equation is called **homogeneous**.

Thus, $\quad x^2 + xy = 0, \quad 2x^2y - 3xy^2 - 4x^3 - 3y^3 = 0.$

When one equation is homogeneous and the other either linear or quadratic we may solve them as follows:

4.
$$6x^2 - 7xy + 2y^2 = 0, \tag{1}$$
$$x^2 - y = 4. \tag{2}$$

Solution: Divide (1) by y^2,
$$6\left(\frac{x}{y}\right)^2 - 7\left(\frac{x}{y}\right) + 2 = 0.$$

Let $\frac{x}{y} = z$,* $\quad 6z^2 - 7z + 2 = 0.$

Solve for z, $\quad z = \frac{1}{2}$ or $\frac{2}{3}$.

Thus $\quad \frac{x}{y} = \frac{1}{2}$ or $\frac{x}{y} = \frac{2}{3}$.

Solve (2) with $x = 2y$ and $2x = 3y$.

When both equations are homogeneous except for a constant term we may solve as follows:

5.
$$x^2 - xy + 2y^2 = 4, \tag{1}$$
$$2x^2 - 3xy - 2y^2 = 6. \tag{2}$$

Solution: Eliminate the constant term by multiplying (1) by 3 and (2) by 2,
$$3x^2 - 3xy + 6y^2 = 12, \tag{3}$$
$$4x^2 - 6xy - 4y^2 = 12 \tag{4}$$

Subtract (3) from (4), $\quad x^2 - 3xy - 10y^2 = 0$

* We observe that $y \neq 0$. For if $y = 0$ were a value that satisfies equation (1), $x = 0$ would correspond. But $(0, 0)$ does not satisfy (2); thus $y = 0$ is not a value that can occur in the solutions of the equations.

Divide by y^2 and let $\frac{x}{y} = z$, where $y \neq 0$,

$$z^2 - 3z - 10 = 0. \tag{5}$$

Factor, $$(z-5)(z+2) = 0.$$

The roots are $$\frac{x}{y} = 5,\ \frac{x}{y} = -2. \tag{6}$$

Solve (6) with (1).

When one equation is quadratic in a binomial expression we may solve as follows:

6. $$x - y - \sqrt{x-y} = 2, \tag{1}$$
$$x^3 - y^3 = 2044. \tag{2}$$

Solution: Let $$\sqrt{x-y} = z.$$

Then (1) becomes $$z^2 - z = 2.$$

Solving for z, $$z = 2 \text{ or } -1.$$

Thus
or $$\left.\begin{aligned} x - y &= 4 \\ x - y &= 1 \end{aligned}\right\} \tag{3}$$

Solve (3) with (2) as in exercise 8.

7. $x^2 + y^2 = xy = x + y.$

8. $x^3 + y^3 = 7xy = 28(x+y).$

9. $x^3 + xy^2 = 2,$
$y^3 + x^2y = 4.$

10. $x^2y = a,$
$xy^2 = b.$

11. $x(y-1) = 10,$
$y(x-1) = 12.$

12. $x^2 + y^2 = a,$
$xy = b.$

13. $x^2y + xy^2 = a,$
$x^2y - xy^2 = b.$

14. $x + xy = 35,$
$y + xy = 32.$

15. $x(x^3 + y^3) = 7,$
$y(x^3 + y^3) = 1.$

16. $2x^2 - 3y^2 = 6,$
$3x^2 - 2y^2 = 19.$

17. $3x^2 - 2y^2 = 9,$
$3x^2 - 2y^2 = 16.$

18. $5x^2 + 2y^2 = 22,$
$3x^2 - 5y^2 = 7.$

19. $x^2 + xy + y^2 = 2,$
$x^2 - xy + y^2 = 6.$

HINT. Eliminate x^2 or y^2 as if the equations were linear equations in x^2 and y^2.

20. $x + xy + y = 5,$
$x^2 + xy + y^2 = 7.$

21. $5x + y + 3 = 2xy,$
$xy = 2x - y + 9.$

22. $x^2 - xy + y^2 = 37,$
$x^2 - y^2 = 40.$

23. $(x+y)(8-x) = 10,$
$(x+y)(5-y) = 20.$

24. $(x^2 + y^2)(x+y) = 5,$
$xy(x+y) = -2.$

25. $(x+y)^2 = 3x^2 - 2,$
$(x-y)^2 = 3y^2 - 11.$

26. $x + \sqrt[3]{x^2 y} = a,$
$y + \sqrt[3]{x y^2} = b.$

27. $x\sqrt{x+y} = 3,$
$y\sqrt{x+y} = 1.$

28. $x + y = 58,$
$\sqrt{x} + \sqrt{y} = 10.$

29. $\sqrt[3]{x} + \sqrt[3]{y} = a,$
$x + y = b.$

30. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = \sqrt[3]{3},$
$x + y = 3.$

31. $\sqrt{x} - \sqrt{y} = 2,$
$(x + y)\sqrt{xy} = 510.$

32. $4x^2 - 9y^2 = 0,$
$4x^2 + y^2 = 8(x + y).$

33. $\sqrt{x-5} + \sqrt{y+2} = 5,$
$x + y = 16.$

34. $3xy - 2(x + y) = 28,$
$2xy - 3(x + y) = 2.$

35. $xy + xy^{-1} = x^2 + y^2,$
$xy - xy^{-1} = 2(x^2 + y^2).$

36. $x^2 + y^2 + x + y = 18,$
$x^2 - y^2 + x - y = 6.$

37. $x^{-2} + 2y^{-2} = 12,$
$x^{-2} - x^{-1}y^{-1} + y^{-2} = 4.$

38. $3x^2 - 2y^2 = 6(x - y),$
$xy = 0.$

39. $x^2 + y^2 - 5(x + y) = 8,$
$x^2 + y^2 - 3(x + y) = 28.$

40. $x^2 - xy + y^2 = 13(x - y),$
$xy = 12.$

41. $2x^2 - 3xy + 5y - 5 = 0,$
$(x - 2)(y - 1) = 0.$

42. $\sqrt{x(1-y)} + \sqrt{y(1-x)} = a,$
$x + y = b.$

43. $\sqrt{5-3x+x^2} + \sqrt{5-3y+y^2} = 6,$
$x + y = 3.$

44. $\sqrt{1-x^2}\,\sqrt{1-y^2} + xy = \frac{77}{85},$
$x - y = \frac{28}{85}.$

45. $\dfrac{1}{x} + \dfrac{1}{y} = 5,$
$x - y = .3.$

46. $\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{16}{15},$
$3x^2 + 5y^2 = 120.$

47. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{25}{12},$
$x^2 - y^2 = 28.$

48. $\dfrac{x\sqrt{x} + y\sqrt{y}}{x\sqrt{x} - y\sqrt{y}} = \dfrac{1}{2},$
$x^3 - 8 = 8 - y^3.$

49. $\sqrt{\dfrac{5x}{x+y}} + \sqrt{\dfrac{x+y}{5x}} = \dfrac{3\sqrt{2}}{2},$
$xy - (x + y) = 1.$

50. $\sqrt{y} - \sqrt{a-x} = \sqrt{y-x},$
$\dfrac{\sqrt{y-x} + \sqrt{a-x}}{\sqrt{a-x}} = \dfrac{5}{2}.$

51. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2,$
$\dfrac{bx + ay}{bx - ay} = \dfrac{m}{n}.$

52. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{2},$
$\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{5}{4}.$

53. $\dfrac{x^3}{y} - \dfrac{y^3}{x} = \dfrac{15}{2},$
$\dfrac{x}{y} - \dfrac{y}{x} = \dfrac{3}{2}.$

HINT. Let $\dfrac{1}{x} = u$, $\dfrac{1}{y} = v$.

54. $x\left(1+\dfrac{x}{y}\right)=2,$
$y\left(1+\dfrac{y}{x}\right)=3.$

55. $\dfrac{x-1}{y-1}=\dfrac{a-1}{b-1},$
$\dfrac{x^3-1}{y^3-1}=\dfrac{a^3-1}{b^3-1}.$

56. $x=10\cdot\dfrac{y-1}{y+1},$
$y=\dfrac{9}{2}\cdot\dfrac{x-1}{x+1}.$

57. $x\sqrt{1-y^2}=\dfrac{180}{221},$
$y\sqrt{1-x^2}=\dfrac{40}{221}.$

58. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{6},$
$\dfrac{1}{x+2}+\dfrac{1}{y+1}=\dfrac{1}{2}.$

59. $\dfrac{4}{x}-\dfrac{3}{y}=1,$
$\dfrac{5}{x+3}-\dfrac{2}{y-1}=2.$

60. $\left(3-\dfrac{6y}{x+y}\right)^2+\left(3+\dfrac{6y}{x-y}\right)^2=82,$
$xy=2.$

PROBLEMS

1. Two numbers are in the ratio 5:3. Their product is 735. What are the numbers?

2. Divide the number 100 into two parts such that the sum of their squares is 5882.

3. The sum of the squares of two numbers increased by the first is 205; if increased by the second the result is 200. What are the numbers?

4. The diagonal of a rectangle is 85 feet long. If each side were longer by 2 feet, the area would be increased by 230 square feet. Find the length of the sides.

5. The diagonal of a rectangle is 89 feet long. If each side of the rectangle were 3 feet shorter the diagonal would be 85 feet long. How long are the sides?

6. The sum of two numbers is 30. If one decreases the first by 3 and the second by 2 the sum of the reciprocals of the diminished numbers is $\frac{1}{6}$. What are the numbers?

7. The sum of the squares of two numbers is 370. If the first were increased by 1 and the second by 3, the sum of the squares would be 500. What are the numbers?

8. A number of persons stop at an inn, and the bill for the entire party is \$24. If there had been 3 more in the party, the bill would have been \$33. How many were in the party and how much did each pay?

9. A fruit seller gets \$2 for his stock of oranges. If his stock had contained 20 more and he had charged $\frac{2}{7}$ of a cent more for each, he would have received \$3 for his stock. How many oranges had he and how much did he get apiece for them?

10. A man has a rectangular plot of ground whose area is 1250 square feet. Its length is twice its breadth. He wishes to divide the plot into a rectangular flower bed, surrounded by a path of uniform breadth, so that the bed and the path may have equal areas. Find the width of the path.

11. In going 7500 yards one of the front wheels of a carriage makes 1000 more revolutions than one of the rear wheels. If the wheels were each a yard greater in circumference, the front wheel would make 625 more revolutions than the rear wheel. What is the circumference of the wheels?

12. A man has \$539 to spend for sheep. He wishes to keep 14 of the flock that he buys, but to sell the remainder at a gain of \$2 per head. This he does and gains \$28. How many sheep did he buy and at what price each?

13. A man buys two kinds of cloth, brown and black. The brown costs 25 cents a yard less than the black, and he gets 2 yards less of it. He spends \$28 for the black cloth and \$25 for the brown. How much was each a yard and how many yards of each did he get?

14. A man left an estate of \$54,000 to be divided among 8 persons, namely, his sons and his nephews. His children together receive twice as much as his nephews, and each one of his children receives \$8400 more than each one of his nephews. How many sons and how many nephews were there?

15. A and B buy cloth. B gives \$9 more for 60 yards than A does for 45 yards; also B gets one yard more for \$9 than A does. How much does each pay?

16. A sum of money and its interest amount to \$22,781 at the end of a year. If the amount had been greater by \$200 and the interest $\frac{1}{4}$ of 1 per cent higher, the amount at the end of the year would have been \$23,045. What was the sum of money and what was the interest?

17. If one divides a number with two digits by the product of its digits, the result is 3. Invert the order of the digits and the resulting number is in the ratio 7 : 4 to the original number. What is the number?

18. What number of two digits is 4 less than the sum of the squares of its digits and 5 greater than twice their product?

19. Increase the numerator of a fraction by 6 and diminish the denominator by 2, and the new fraction is twice as great as the original fraction. Increase the numerator by 3 and decrease the denominator by the same, and the fraction goes into its reciprocal. What is the fraction?

121. Equivalence of pairs of equations. In the theorems of this section the capital letters represent polynomials in x and y, and the small letters represent numbers not equal to zero.

THEOREM I. *The pairs of equations*

$$\left.\begin{aligned} A &= a \\ B^2 &= b^2 \end{aligned}\right\} (1) \qquad \left.\begin{aligned} A &= a \\ B &= \pm b \end{aligned}\right\} (2)$$

are equivalent.

If (x_1, y_1) be a pair of values that satisfy (1), then when x and y in B^2 are replaced by x_1 and y_1 the equation $B^2 = b^2$ is a numerical identity. These values (x_1, y_1) must then satisfy one of the equations $B = \pm b$, for if they did not, but only satisfied the equation say $B = c$ when $c \neq \pm b$, then the hypothesis that $B^2 = b^2$ is satisfied by (x_1, y_1) would be contradicted.

Conversely, any pair of values that satisfy $B = \pm b$ evidently satisfy $B^2 = b^2$.

This theorem is used, for instance, in exercise 2, p. 114, and justifies the assumption that

$$\left.\begin{aligned} x^2 + xy &= 6 \\ x^2 + 2xy + y^2 &= 16 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x^2 + xy &= 6 \\ x + y &= \pm 4 \end{aligned}\right\}$$

are equivalent pairs of equations.

THEOREM II. *The pairs of equations*

$$\left.\begin{aligned} A &= a \\ AB &= ab \end{aligned}\right\} (1) \quad \textit{and} \quad \left.\begin{aligned} A &= a \\ B &= b \end{aligned}\right\} (2)$$

are equivalent.

If $A = a$ and $B = b$ are satisfied by a pair of numbers (x_1, y_1), we multiply the identities and obtain $AB = ab$.

Conversely, if $A = a$, $AB = ab$ are identically satisfied by a pair (x_1, y_1), since $a \neq 0$ we can divide the second identity by the first and obtain $B = b$. Thus if (x_1, y_1) satisfy one pair of equations they satisfy the other pair.

This theorem is assumed in exercise 3, p. 115, to show that

$$\left.\begin{aligned} x^3 + y^3 &= 12 \\ x + y &= 2 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x^2 - xy + y^2 &= 6 \\ x + y &= 2 \end{aligned}\right\} \text{ are equivalent.}$$

THEOREM III. *The pairs of equations*

$$\left.\begin{matrix} A=0 \\ B=0 \end{matrix}\right\}(1) \qquad \left.\begin{matrix} aA+bB=0 \\ cA+dB=0 \end{matrix}\right\}(2)$$

are equivalent where a, b, c, and d are numbers such that

$$ad - bc \neq 0.$$

If (x_1, y_1) satisfy (1), evidently it also satisfies (2). Thus all solutions of (1) are among those of (2).

Conversely, if (x_1, y_1) satisfy (2), then

$$A = -\frac{bB}{a} = -\frac{dB}{c}.$$

Thus $$(ad - bc)B = 0.$$

Thus since $(ad - bc) \neq 0$,

$$B = 0.$$

Similarly, $$A = 0.$$

This theorem has been assumed in exercises 1, 2, 3, 6, p. 114. In 1, for example, it is necessary to show that

$$\left.\begin{matrix} x^2+y^2=17 \\ xy=4 \end{matrix}\right\}(1) \quad \text{and} \quad \left.\begin{matrix} x^2+2xy+y^2=25 \\ x^2-2xy+y^2=9 \end{matrix}\right\}(2)$$

are equivalent. In this case $a = c = 1$, $b = -d = 2$. Thus $ad - bc = -4 \neq 0$.

122. Incompatible equations. When a pair of simultaneous equations can be proven equivalent to a pair of equations which contradict each other or are absurd, they are incompatible and have no finite solution.

EXAMPLE 1.

$$\begin{aligned} xy &= 1 \\ xy &= -1 \end{aligned}$$

Subtract, $$0 = 2$$

EXAMPLE 2.

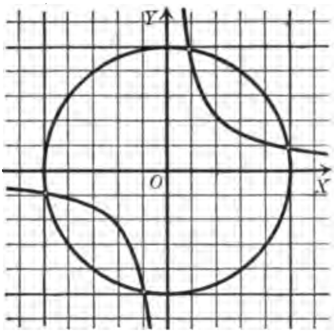
$$x^2 + y^2 = 4, \tag{1}$$
$$4x^2 + 4y^2 = 49. \tag{2}$$

Multiply (1) by 4,

$$\begin{aligned} 4x^2 + 4y^2 &= 16 \\ 4x^2 + 4y^2 &= 49 \end{aligned}$$

Subtract, $$0 = 33$$

123. Graphical representation of simultaneous quadratic equations. Every equation that we have considered may be represented graphically by plotting in accordance with the method already given (p. 93).



The solution of simultaneous equations is represented by the points of intersection of the corresponding graphs.

Thus the equations

$$x^2 + y^2 = 25,$$
$$2xy = 9$$

have the solutions

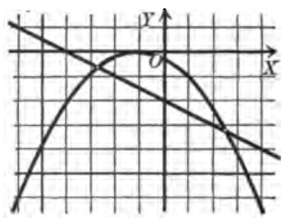
$$x = \pm 2 \pm \frac{\sqrt{34}}{2}, \quad y = \mp 2 \pm \frac{\sqrt{34}}{2},$$

or
$$x = 4.9, .9, -.9, -4.9,$$
$$y = .9, 4.9, -4.9, -.9.$$

These equations have as their graph the preceding figure.

The equations

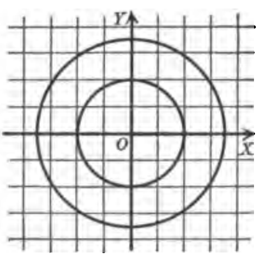
$$x^2 + 2x + 4y + 1 = 0,$$
$$x + 2y + 4 = 0,$$



which have the solutions

$$x = \pm \sqrt{7} = \pm 2.6,$$
$$y = -2 \mp \sqrt{\tfrac{7}{2}} = -3.3 \text{ or } -.7,$$

have as their graph the figure shown above.



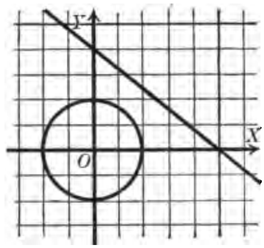
As in the case of linear equations, incompatible equations afford graphs which do not intersect. Thus the graph of the equations in example 2, p. 121, is found to be two concentric circles, as is shown in the adjacent figure.

Simultaneous equations which have imaginary solutions also lead to non-intersecting graphs (p. 101).

Thus the equations

$$x^2 + y^2 = 4,$$
$$4x + 5y = 20$$

have the adjacent figure as their graph.



EXERCISES

1. Interpret the graphical meaning of equivalent pairs of equations.

2. Plot and solve $x^2 + y^2 = 2,$
$x + y = 2.$

What general statement concerning the graphical meaning of a *single solution* of quadratic and linear equations does this example suggest?

3. Plot and solve the following:

(a) $x^2 + y^2 = 25,$
$4x^2 + 9y^2 = 144.$

(b) $x^2 + y^2 = 25,$
$4x^2 - 8x + 9y^2 = 140.$

(c) $x^2 + y^2 = 25,$
$5x^2 + y^2 = 25.$

What general statement concerning the graphical interpretation of four, three, or two real solutions of equations do these examples suggest?

4. State the *algebraical* condition under which two quadratic equations have four, three, two, or one real solutions (see p. 113).

5. Plot and solve the following:

(a) $x^2 + y = 0,$
$x^2 - 3y = 0.$

(b) $x^2 + y^2 = 9,$
$x^2 - y^2 = 0.$

(c) $4x - 2y = 3,$
$xy - y = 0.$

(d) $xy = 1,$
$x^2 + y^2 = 16.$

(e) $4x^2 + 9y^2 = 36,$
$x^2 = -4y.$

(f) $xy = 1,$
$2x - 3y = 18.$

124. Graphical meaning of homogeneous equations. Consider for example the homogeneous equation

$$3x^2 - 10xy - 8y^2 = 0. \qquad (1)$$

If we let $z = \frac{y}{x}$, we get $3 - 10z - 8z^2 = 0,$

or $8z^2 + 10z - 3 = 0,$

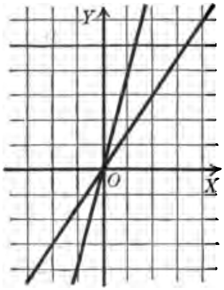
or $(4z - 1)(2z + 3) = 0.$

The roots are $\qquad z = \frac{1}{4}$ and $z = -\frac{3}{2}$.

Thus $\qquad \frac{y}{x} = \frac{1}{4}$ and $\frac{y}{x} = -\frac{3}{2}$,

or $\qquad 4y - x = 0$ and $3x + 2y = 0$.

These equations represent two straight lines through the origin which taken together form the graph of equation (1). This example may obviously be generalized: *Any homogeneous equation of the form $ax^2 + bxy + cy^2 = 0$ with positive discriminant represents two straight lines through the origin.* Such an equation is equivalent to two linear equations.



In an example like 5, p. 115, we obtain in place of the given pair of equations a pair of equivalent equations one of which is homogeneous and the other of which is factorable. We can learn the graphical meaning of this method of solution by studying a particular case. Consider for example the equations:

$$x^2 + 2xy + 7y^2 = 24, \qquad (1)$$

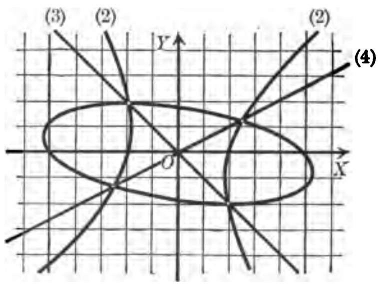
$$2x^2 - xy - y^2 = 8. \qquad (2)$$

By eliminating the constant terms we obtain the product of the two equations $x + y = 0$ and $x - 2y = 0$. Thus the problem of solving (1) and (2) is replaced by that of solving the two following pairs of equations:

$$x^2 + 2xy + 7y^2 = 24,$$
$$x + y = 0,$$

or

$$x^2 + 2xy + 7y^2 = 24,$$
$$x - 2y = 0.$$



The graphical meaning of this method of solving the equations (1) and (2) is seen in the fact that the problem of finding the points of intersection of the graph of equation (1) with that of (2) is changed to that of finding the intersection of the graph of (1) with a pair of straight lines. This appears in the figure where the curves and lines are numbered as above. The closed curve represents (1).

CHAPTER XI

MATHEMATICAL INDUCTION

125. General statement. Many theorems are capable of direct and simple proof in special cases, while for the general case a direct proof is difficult and complicated.

If we ask whether $x^n - 1$ is divisible by $x - 1$, it is easy to make the actual division for any particular value of n, as $n = 2$ or $n = 3$. But if $x^3 - 1$ is shown divisible by $x - 1$, we are no wiser than before concerning the divisibility of $x^4 - 1$. Suppose, however, we can prove that the divisibility for $n = m + 1$ follows from that for $n = m$, whatever value m may have. Then since we have established the fact by direct division for $n = 3$, we may be assured of the divisibility for $n = 4$, then for $n = 5$, and so on.

Now $$x^{m+1} - 1 = x(x^m - 1) + (x - 1)$$

is identically true. If $x - 1$ is a factor of $x^m - 1$ for a given value of m, it is a factor of the right-hand member and consequently a factor of the left-hand member (§ 69), which was to be proved. Thus the divisibility of $x^m - 1$ by $x - 1$ is established for any integral value of n greater than the one for which the division has actually been carried out.

To complete the proof of a theorem by **mathematical induction** we must make two distinct steps.

First. Establish the theorem for some particular case or cases, preferably for $n = 1$ and $n = 2$.

Second. Show that the theorem for $n = m + 1$ follows from its assumed validity for $n = m$.

Example. Prove that the sum of the cubes of the integers from 1 to n is $$\{\tfrac{1}{2}[n(n+1)]\}^2.$$

To prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \{\tfrac{1}{2}[n(n+1)]\}^2$.

First. This theorem is true for $n = 1$.

For $$1^3 = 1 = \{\tfrac{1}{2}[1(2)]\}^2 = 1^2 = 1.$$

The theorem is also true for $n = 2$.

For $$1^3 + 2^3 = 9 = \{\tfrac{1}{2}[2(2+1)]\}^2 = (\tfrac{1}{2}\cdot 6)^2 = 3^2 = 9.$$

Second. Assume the theorem for $n = m$,*

$$1^3 + 2^3 + \cdots + m^3 = \{\tfrac{1}{2}[m(m+1)]\}^2.$$

Add $(m+1)^3$ to both sides of the equation,

$$\begin{aligned} 1^3 + 2^3 + \cdots + m^3 + (m+1)^3 &= \{\tfrac{1}{2}[m(m+1)]\}^2 + (m+1)^3 \\ &= [(\tfrac{1}{2}m)^2 + m + 1](m+1)^2 \\ &= \left(\frac{m^2 + 4m + 4}{4}\right)(m+1)^2 \\ &= [\tfrac{1}{2}(m+1)(m+2)]^2, \end{aligned}$$

which is the form desired, i.e. $m + 1$ replaces m in the formula.

EXERCISES

Prove by mathematical induction that

1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
2. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.
3. $3 + 6 + 9 + \cdots + 3n = \frac{3}{2}n(n+1)$.
4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
5. $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.
6. $4^2 + 7^2 + 10^2 + \cdots + (3n+1)^2 = \frac{1}{2}n(6n^2 + 15n + 11)$.
7. $x^n - y^n$ is divisible by $x - y$ for any integral values of n.
8. $x^{2n} - y^{2n}$ is divisible by $x + y$ for any integral values of n.
9. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + \cdots + n\cdot(n+1) = \frac{1}{3}n(n+1)(n+2)$.
10. $1\cdot 1 + 2\cdot 3^2 + 3\cdot 5^2 + \cdots + n(2n-1)^2 = \frac{1}{6}n(n+1)(6n^2 - 2n - 1)$.
11. $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$.
12. $(1^3 + 2^3 + 3^3 + \cdots + n^3) + 3(1^5 + 2^5 + 3^5 + \cdots + n^5)$
    $= 4(1 + 2 + 3 + \cdots + n)^3$.

* This statement does not imply that we assume the validity of the formula for any values for which it has not yet been established, but only for values of m not greater than 2.

13. $1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + \cdots + (2n-1)3^n = 3^{n+1}(n-1) + 3.$

14. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n \cdot (n+1)} = \dfrac{n}{n+1}.$

15. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}.$

16. $2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + (n+1)(n+4) = \dfrac{n(n+4)(n+5)}{3}.$

17. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2n(2n+2) = \dfrac{n}{3}(2n+2)(2n+4)$

18. A pyramid of shot stands on a triangular base having m shot on a side. How many shot are in the pile?

CHAPTER XII

BINOMIAL THEOREM

126. Statement of the binomial theorem. When in previous problems any power of a binomial has been required we have obtained the result by direct multiplication. We can, however, deduce a general law known as the **binomial theorem**, which gives the form of development of $(a + b)^n$, where n is any positive integer and a and b are any algebraical or arithmetical expressions. This law is as follows:

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n\cdot(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots + b^n.$$

From this expression we deduce the following

RULE FOR THE DEVELOPMENT OF $(a + b)^n$.

The first term is a^n.

The second term is $\frac{n}{1}a^{n-1}b$.

To obtain any term from the preceding term, decrease the exponent of a in the preceding term by 1 and increase the exponent of b by 1 for the new exponents. Multiply the coefficient of the preceding term by the exponent of a, and divide it by the exponent of b increased by 1 for the new coefficient.

REMARK. In practice it is usually more convenient first to write down all the terms with their proper exponents, and then form the successive coefficients.

EXERCISES

Verify by multiplication the rule given for the following:

1. $(a+b)^3$.
2. $(x-y)^3$.
3. $(2a+3b)^4$.
4. $(\sqrt{x}+\sqrt{y})^8$.
5. $(2a-b)^4$.
6. $(x-\sqrt{y})^4$.
7. $(3a-2b)^3$.
8. $(a^{-1}x+b^{-1}y)^4$.

127. Proof of the binomial theorem. We have already stated that

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots \qquad (1)$$

and have seen that it is justified for every particular case that we have tested. By complete induction we now prove this theorem when n is a positive integer.

First. Let $n = 2$.

That is, $$(a+b)^2 = a^2 + 2\,ab + b^2.$$

This expression evidently obeys the law as stated in (1).

Second. Assume the theorem for $n = m$.

That is, $$(a+b)^m = a^m + \frac{m}{1}a^{m-1}b + \frac{m(m-1)}{1 \cdot 2}a^{m-2}b^2 + \cdots. \qquad (2)$$

Multiply both members by $a + b$,

$$\begin{aligned}(a+b)^{m+1} &= a^{m+1} + \frac{m}{1}a^m b + \frac{m}{1}a^{m-1}b^2 \\ &\qquad + a^m b + \frac{m(m-1)}{2}a^{m-1}b^2 + \cdots \\ &= a^{m+1} + (m+1)a^m b + \frac{(m+1)m}{2}a^{m-1}b^2 + \cdots.\end{aligned}$$

This expression is identical with (2) except that $(m+1)$ replaces m. Hence the theorem is established so far as the first three terms are concerned.

128. General term. Though we have stated the binomial theorem for a general value of n, we have only established the exact form of the first three terms.

Let $(a+b)^n = a^n + c_1 a^{n-1}b + c_2 a^{n-2}b^2 + \cdots$.

We note that the sum of the exponents of a and b is n in any term of the development of $(a+b)^n$. Also the exponent of b in the $(r+1)$st term is r.

We have already seen that

$$c_1 = \frac{n}{1},\ c_2 = \frac{n(n-1)}{1 \cdot 2},$$

and that the first three terms are

$$a^n,\ \frac{n}{1}a^{n-1}b,\ \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$$

respectively. This *indicates* that the coefficient of the next term will be $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$ and in general that the coefficient of the $(r+1)$st term has the form

$$c_r = \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}, \tag{1}$$

which is in fact the form that our rule (§ 126) would afford for any *particular* value of r.

This affords the following

RULE. *The $(r+1)$st term of $(a+b)^n$ is*

$$\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}a^{n-r}b^r.$$

The form of the coefficient may be easily remembered since the denominator consists of the product of the integers from 1 to r, while the numerator contains an equal number of factors consisting of descending integers beginning with n.

For any particular values of n and r we could easily verify the rule by direct multiplication. For the rigorous proof see p. 178.

EXERCISES

Develop by the binomial theorem:

1. $\left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a^2}\right)^6$.

Solution:

$$\left(\frac{a}{\sqrt{x}}\right)^6 + \frac{6}{1}\left(\frac{a}{\sqrt{x}}\right)^5\left(-\frac{\sqrt{x}}{a^2}\right)^1 + \frac{6\cdot 5}{1\cdot 2}\left(\frac{a}{\sqrt{x}}\right)^4\left(-\frac{\sqrt{x}}{a^2}\right)^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}\left(\frac{a}{\sqrt{x}}\right)^3\left(-\frac{\sqrt{x}}{a^2}\right)^3$$
$$+ \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}\left(\frac{a}{\sqrt{x}}\right)^2\left(-\frac{\sqrt{x}}{a^2}\right)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}\left(\frac{a}{\sqrt{x}}\right)^1\left(-\frac{\sqrt{x}}{a^2}\right)^5$$
$$+ \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(-\frac{\sqrt{x}}{a^2}\right)^6$$
$$= \frac{a^6}{x^3} - 6\frac{a^3}{x^2} + \frac{15}{x} - \frac{20}{a^3} + \frac{15x}{a^6} - \frac{6x^2}{a^9} + \frac{x^3}{a^{12}}.$$

2. $(\frac{3}{4} - \frac{4}{3}a)^6$.

3. $(\sqrt[4]{a} + \sqrt[3]{b})^5$.

4. $\left(a + \frac{1}{a}\right)^6$.

5. $\left(\frac{x}{y} - \frac{y}{x}\right)^7$.

6. $\left(\frac{2x}{3y}-\frac{3y}{2x}\right)^5$.

7. $\left(\frac{x^2}{3}-\frac{2}{x^3}\right)^5$.

8. $\left(\frac{\sqrt{x}}{10}-\frac{3\sqrt{y}}{5}\right)^4$.

9. $\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{3}\right)^6$.

10. $(1+\sqrt{a})^7-(1-\sqrt{a})^7$.

11. $(\sqrt{x}+\sqrt{y})^4+(\sqrt{x}-\sqrt{y})^4$.

12. Find the 8th term in the development of $\left(\frac{2x}{3y}+\frac{3y}{2x}\right)^{10}$.

Solution: $n=10,\ r+1=8.$

The $(r+1)$st term of $(a+b)^n$ is (§ 128)

$$\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}a^{n-r}b^r.$$

In this case we get

$$\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\left(\frac{2x}{3y}\right)^3\left(\frac{3y}{2x}\right)^7$$
$$=120\cdot\frac{2^3x^3}{3^3y^3}\cdot\frac{3^7y^7}{2^7x^7}=120\cdot\frac{3^4y^4}{2^4x^4}$$
$$=\frac{120\cdot 81}{16}\cdot\frac{y^4}{x^4}=\frac{1215y^4}{2x^4}.$$

13. Find the 7th term of $\left(a+\frac{1}{a}\right)^{18}$.

14. Find the 6th term of $\left(\frac{x}{2y}-\frac{2y}{x}\right)^{18}$.

15. Find the 8th term of $\left(\frac{\sqrt{x}}{y}-\frac{\sqrt{y}}{x}\right)^{15}$.

16. Find the 6th term of $\left(2a\sqrt{b}-\frac{1}{2a\sqrt{b}}\right)^{13}$.

17. Find the 7th term of $\left(\frac{\sqrt{a}}{\sqrt[3]{b}}-\frac{\sqrt{b}}{\sqrt[3]{a}}\right)^{15}$.

18. Find correct to three decimal places $(.9)^8$.

Solution: $(.9)^8=(1-.1)^8$

$$=(1)^8-\frac{8}{1}(1)^7(.1)+\frac{8\cdot 7}{1\cdot 2}(1)^6(.1)^2-\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}(1)^5(.1)^3$$
$$+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}(1)^4(.1)^4+\cdots$$
$$=1-8\cdot 0.1+28\cdot 0.01-56\cdot 0.001+70\cdot 0.0001$$
$$=1+0.28+0.0070-0.8-0.056$$
$$=1.2870-0.856=.431.$$

In this exercise any terms beyond those taken would not affect the first three places in the result.

Compute the following correct to three places:

19. $(1.1)^{10}$. **20.** $(2.9)^{8}$. **21.** $(.98)^{11}$. **22.** $(1.01)^{6}$.

23. $(\frac{49}{50})^{8}$. **24.** $(\frac{41}{40})^{10}$. **25.** $(98)^{6}$. **26.** $(203)^{5}$.

27. In what term of $(a+b)^{20}$ does a term involving a^{14} occur?

28. For what kind of exponent may a and b enter the same term with equal exponents?

29. For what kind of exponent is the number of terms in the binomial development even?

30. Find the first three and the last three terms in the development of

$$\left(\sqrt[3]{ab^2} - 2\frac{1}{a^2b^{-\frac{1}{2}}}\right)^{24}.$$

CHAPTER XIII

ARITHMETICAL PROGRESSION

129. Definitions. A series of numbers such that each number minus the preceding one always gives the same positive or negative number is called an **arithmetical series** or **arithmetical progression** (denoted by A.P.).

The constant difference between any term and the preceding term of an A.P. is called the **common difference.**

The series 4, 7, 10, 13, ⋯ is an A.P. with the common difference 3. The series 8, $6\frac{1}{2}$, 5, $3\frac{1}{2}$, ⋯ is an A.P. with the common difference $-\frac{3}{2}$. The series 4, 6, 7, 9, 10, ⋯ is not an A.P.

EXERCISES

Determine whether the following series are in A.P. If so, find the common difference.

1. $6,\ 3\frac{11}{16},\ 1\frac{5}{8},\ \cdots.$

2. $27,\ 22\frac{1}{2},\ 18,\ \cdots.$

3. $6,\ 4\frac{1}{2},\ 3,\ 1\frac{1}{2},\ \cdots.$

4. $5,\ -2,\ -8,\ \cdots.$

5. $\sqrt{\frac{1}{2}},\ \sqrt{2},\ 3\sqrt{\frac{1}{2}},\ \cdots.$

6. $8,\ 5\frac{7}{8},\ 3\frac{3}{4},\ 1\frac{5}{8},\ \cdots.$

7. $\frac{1}{\sqrt{2}},\ \frac{2}{\sqrt{2}},\ \frac{4}{\sqrt{2}},\ \cdots.$

8. $\frac{\sqrt{2}-1}{2},\ \frac{\sqrt{2}}{2},\ \frac{1}{2(\sqrt{2}-1)},\ \cdots.$

9. $3,\ -\frac{1}{7},\ -3\frac{2}{7},\ -6\frac{3}{7},\ \cdots.$

10. $\frac{\sqrt{3}}{2},\ \frac{3\sqrt{3}+2}{6},\ \frac{-39}{6(\sqrt{3}-4)},\ \cdots.$

130. The *n*th term. The terms of an A.P. in which a is the first term and d the common difference are as follows:

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots. \qquad (1)$$

The multiple of d is seen to be 1 in the second term, 2 in the third term, and in fact always one less than the number of the term. If we call l the nth term, we have

$$l = a + (n-1)d.$$

We may also write the series in which l is the nth term as follows:

$$a,\ a+d,\ a+2d,\ \cdots,\ l-2d,\ l-d,\ l.$$

131. The sum of the series. We may obtain a formula for computing the sum of the first n terms of an A.P. by the following

THEOREM. *The sum s of the first n terms of the series $a,\ a+d,\ \cdots,\ l-d,\ l$ is*

$$s=\frac{n}{2}(a+l).$$

By definition,

$$s=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l. \quad (1)$$

Inverting the order of the terms of the right-hand member,

$$s=l+(l-d)+(l-2d)+\cdots+(a-2d)+(a+d)+a. \quad (2)$$

Adding (1) and (2) term by term,

$$2s=(l+a)+(l+a)+(l+a)+\cdots+(l+a)+(l+a)+(l+a)$$
$$=n(a+l).$$

Thus $$s=\frac{n}{2}(a+l).$$

132. Arithmetical means. The terms of an A.P. between a given term and a subsequent term are called **arithmetical means** between those terms. By *the* arithmetical mean of two numbers is meant the number which is the second term of an arithmetical series of which they are the first and third terms. Thus the arithmetical mean of two numbers a and b is $\frac{a+b}{2}$, since the numbers $a,\ \frac{a+b}{2},\ b$ are in A.P. with the common difference $\frac{b}{2}$.

The two formulas

$$l=a+(n-1)d, \quad \text{(I)}$$

$$s=\frac{n}{2}(a+l) \quad \text{(II)}$$

contain the **elements** $a,\ l,\ s,\ n,\ d$. Evidently when any three are known the remaining two may be found by solving the two equations (I) and (II).

EXERCISES

1. Find the 16th term and the sum of the series $4, 2, 0, -2, \cdots$.

Solution: $n = 16,\ a = 4,\ d = 2 - 4 = -2.$

$$l = a + (n-1)d = 4 + 15(-2) = -26,$$

$$s = \frac{n}{2}(a + l) = \frac{16}{2}(4 - 26) = -176.$$

2. $l = 42$, $a = -3$, $d = 3$. Find n and s.
3. $a = -4$, $n = 8$, $s = 64$. Find d and l.
4. $d = -\frac{1}{2}$, $n = 6$, $l = 21$. Find s and a.
5. $d = -\frac{1}{3}$, $n = 10$, $s = 65$. Find a and l.
6. $s = 16\frac{1}{2}$, $l = 4$, $a = -3$. Find d and n.
7. $l = 22$, $s = 243$, $n = 13$. Find a and d.
8. $s = -15$, $l = -2$, $d = 2$. Find n and a.
9. $d = 4\frac{1}{2}$, $a = -16$, $s = 140$. Find n and l.
10. Insert 8 arithmetical means between 4 and 28.
11. Find expressions for n and s in terms of a, l, and d.
12. Find expressions for l and a in terms of s, n, and d.
13. Find expressions for a and s in terms of d, l, and n.
14. Find expressions for d and n in terms of s, a, and l.
15. Find the 13th term and the sum of the series

$$\frac{\sqrt{2}-1}{2},\ \frac{\sqrt{2}}{2},\ \frac{1}{2(\sqrt{2}-1)},\ \cdots.$$

16. Find the 10th term and the sum of the series

$$\frac{\sqrt{3}}{2},\ \frac{3\sqrt{3}+2}{6},\ \frac{\sqrt{3}}{2}+\frac{2}{3},\ \cdots.$$

17. Insert 4 arithmetical means between $\frac{1}{\sqrt{2}}$ and $\frac{5\sqrt{2}}{2}$.
18. Insert 5 arithmetical means between $\sqrt{\frac{2}{3}}$ and $\frac{10\sqrt{6}}{3}$.
19. Insert 3 arithmetical means between $\frac{\sqrt{3}}{2}$ and $\frac{8+3\sqrt{3}}{6}$
20. Find the 21st term and the sum of the series $\frac{1}{\sqrt{2}},\ \sqrt{2},\ \frac{3\sqrt{2}}{2},\ \cdots$.
21. Find the 10th term and the sum of the series $\frac{\sqrt{2}}{\sqrt{3}},\ \frac{5}{\sqrt{6}},\ \frac{4\sqrt{2}}{\sqrt{3}},\ \cdots$.

22. Find expressions for d and a in terms of s, l, and n.

23. Find expressions for d and l in terms of a, n, and s.

24. Find the 8th term and the sum of the series $x, 4x, 7x, \cdots$.

25. Find the 9th term and the sum of the series $8, 9\frac{1}{3}, 10\frac{2}{3}, \cdots$.

26. Find the 12th term and the sum of the series $8, 7\frac{1}{16}, 6\frac{1}{8}, \cdots$.

27. Find the 8th term and the sum of the series $-8, -4, 0, \cdots$.

28. Find the 12th term and the sum of the series $27, 22\frac{1}{2}, 18, \cdots$.

29. Find the 20th term and the sum of the series $1, -2\frac{1}{2}, -6, \cdots$.

30. Find the 11th term and the sum of the series $5, -3, -11, \cdots$.

31. Find the 9th term and the sum of the series $x - y, x, x + y, \cdots$.

32. Insert $n - 2$ arithmetical means between a and l. Write the first three.

REMARK. Often an exercise may be solved more simply if instead of assuming the series $x, x + 2y, x + 3y, \cdots$ we assume $x - y, x, x + y$ when three terms are required, or $x - 2y, x - y, x, x + y, x + 2y$ when five terms are required, or $x - 3y, x - y, x + y, x + 3y$ when four terms are required.

33. The sum of the first three terms of an A.P. is 15. The sum of their squares is 83. Find the sum of the series to ten terms.

34. Find expressions for n and a in terms of s, l, and d. For what real values of s, l, and d does a series with real terms not exist?

35. In an A.P. where a is the first term and s is the sum of the first n terms, find the expression for the sum of the first m terms.

36. Find expressions for n and l in terms of a, s, and d. For what real values of a, s, and d does a series with real terms not exist?

37. If each term of the series (1), § 130, is multiplied by m, is the new series in A.P., and if so, what are the elements of the new series?

38. If each term of the series (1) in § 130 is increased by b, is the new series in A.P., and if so, what are the elements of the new series?

39. The difference between the third and sixth terms of an A.P. is 12. The sum of the first 10 terms is 45. Find the elements of the series.

40. Find the 10th term of an A.P. whose first and sixteenth terms are 3 and 48. Find also the sum of those eight terms of the series the last of which is 60.

41. Two A.P.'s have the same common difference, and their first terms are 2 and 4 respectively. The sum of the first seven terms of one are to the sum of the first seven terms of the other as 4 is to 5. Find the elements of both series.

42. The three digits of a number are in A.P. The number itself divided by the sum of the digits is 48. The number formed by the same digits in reverse order is 396 less than the original number. What is the number?

CHAPTER XIV

GEOMETRICAL PROGRESSION

133. Definitions. A series of numbers such that the quotient of any term of the series by the preceding term is always the same is called a **geometrical progression** (denoted by G.P.).

The constant quotient of any term by the preceding term of a G.P. is called the **ratio.**

The G.P. series 4, 8, 16, $\cdots$ has the ratio 2. The G.P. series 8, $4\sqrt{2}$, 4, $\cdots$ has the ratio $\frac{\sqrt{2}}{2}$.

EXERCISES

Determine which of the following series are in G.P. and find the ratio.

1. $4,\ 2,\ 1,\ \cdots.$

2. $4,\ 8,\ 16,\ \cdots.$

3. $8,\ -2,\ .5,\ \cdots.$

4. $8,\ -4,\ -2,\ \cdots.$

5. $\sqrt{\frac{2}{3}},\ 1,\ \sqrt{\frac{3}{2}},\ \cdots.$

6. $6,\ -21,\ 73\frac{1}{2},\ \cdots.$

7. $\frac{1}{\sqrt{2}},\ \sqrt{2},\ 2,\ \cdots.$

8. $\frac{1}{\sqrt{2}},\ -2,\ \frac{8}{\sqrt{2}},\ \cdots.$

9. $\sqrt{\frac{3}{5}},\ \frac{1}{\sqrt{5}},\ \frac{1}{\sqrt{15}},\ \cdots.$

10. $\frac{\sqrt{3}}{8},\ \sqrt{\frac{3}{32}},\ \frac{\sqrt{3}}{4},\ \cdots.$

11. $\frac{1}{\sqrt{3}-\sqrt{2}},\ 5-2\sqrt{6},\ 3\sqrt{3}-\sqrt{2},\ \cdots.$

12. $\sqrt{2}-1,\ 1,\ \sqrt{2}+1,\ \cdots.$

134. The nth term. The terms of a G.P. in which a is the first term and r the ratio are as follows:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots.$$

The power of r in the second term is 1, in the third term is 2, and in fact is always one less than the number of the term. If we call l the nth term, we have the following expression for the nth term:

$$l = ar^{n-1}. \qquad \text{(I)}$$

135. The sum of the series. We obtain a formula for finding the sum of the first n terms of a G.P. by the following

THEOREM. *The sum s of the first n terms of the geometrical progression $a, ar, ar^2, \dots$ is*

$$s = \frac{a - rl}{1 - r}.$$

By definition,
$$\begin{aligned} s &= a + ar + ar^2 + \cdots + ar^{n-1} \\ &= a(1 + r + r^2 + \cdots + r^{n-1}) \\ &= a\left(\frac{1 - r^n}{1 - r}\right) && \text{by (III), p. 15} \\ &= \frac{a - rar^{n-1}}{1 - r} = \frac{a - rl}{1 - r}. && \text{by (I), p. 137} \end{aligned}$$

136. Geometrical means. The $n - 2$ terms between the first and the nth term of a G.P. are called the **geometrical means** between those terms.

If one geometrical mean is inserted between two numbers, it is called *the* geometrical mean of those numbers. Thus the geometrical mean between a and b is $\sqrt{ab}$.

The two fundamental formulas

$$l = ar^{n-1}, \tag{I}$$

$$s = \frac{a(1 - r^n)}{1 - r} = \frac{a - rl}{1 - r} \tag{II}$$

contain the five **elements** a, l, r, n, s, any two of which may be found if the remaining three are given.

EXERCISES

1. Find the 7th term and the sum of the G.P. 1, 4, 16, $\cdots$.

Solution: $a = 1,\ n = 7,\ r = 4.$

Substituting in (I), $l = ar^{n-1} = 1 \cdot 4^6 = 4096,$

$$s = \frac{a - rl}{1 - r}.$$

Substituting in (II), $s = \dfrac{1 - 4 \cdot 4096}{1 - 4} = \dfrac{16383}{3} = 5461.$

2. Insert 2 geometrical means between 4 and 32.

3. Insert 4 geometrical means between 32 and 1.

4. Insert 3 geometrical means between 3 and $\frac{16}{27}$.

5. Insert 4 geometrical means between a^5 and b^5.

6. Insert 4 geometrical means between 1 and $9\sqrt{3}$.

7. Insert 3 geometrical means between $\frac{12}{7}$ and $73\frac{1}{2}$.

8. What is the geometrical mean between 3 and 27?

9. Insert 3 geometrical means between $\sqrt[6]{3}$ and $\sqrt[6]{24}$.

10. Insert 4 geometrical means between a and $a^2\sqrt{ab^5}$.

11. Insert 3 geometrical means between $-\frac{4}{9}$ and $-2\frac{1}{4}$.

12. What is the geometrical mean between -2 and $-\frac{8}{9}$?

13. Find the 7th term and the sum of the series $1, 3, 9, \cdots$.

14. Find the 6th term and the sum of the series $2, 4, 8, \cdots$.

15. What is the geometrical mean between $\sqrt[4]{a^3b}$ and $\sqrt[4]{ab^3}$?

16. Find the 7th term and the sum of the series $8, 2, .5, \cdots$.

17. Find the 8th term and the sum of the series $\frac{1}{64}, \frac{1}{16}, \frac{1}{4}, \cdots$.

18. Find the 7th term and the sum of the series $\sqrt[3]{4}, 2, 2\sqrt[3]{2}, \cdots$.

19. Find the 7th term and the sum of the series $\sqrt[3]{2}, \sqrt{2}, \sqrt[3]{4}, \cdots$.

20. Find the 10th term and the sum of the series $\frac{1}{256}, \frac{1}{128}, \frac{1}{64}, \cdots$.

21. Find the 5th term and the sum of the series $\sqrt{2}-1, 1, 1+\sqrt{2}, \cdots$.

22. The first and sixth terms of a G.P. are 1 and 243. Find the intermediate terms.

23. Find the 5th term and the sum of the series

$$\frac{1}{\sqrt{3}+\sqrt{2}},\ 5-2\sqrt{6},\ 9\sqrt{3}-11\sqrt{2}, \cdots.$$

24. Insert 3 geometrical means between $\frac{1}{\sqrt{3}}$ and $\frac{1}{9}$.

25. What is the geometrical mean between $\frac{x-y}{x+y}$ and $\frac{(x+y)^2}{x-y}$?

26. Find the 6th term and the sum of the series $\frac{1}{\sqrt{2}}, -\sqrt{2}, \frac{4}{\sqrt{2}}, \cdots$.

27. Find the 6th term and the sum of the series $\sqrt{\frac{2}{3}}, 1, \frac{\sqrt{3}}{\sqrt{2}}, \cdots$.

28. Find the 5th term and the sum of the series $\sqrt{2}, -1, \frac{\sqrt{2}}{2}, \cdots$.

29. Find the 6th term and the sum of the series $\frac{\sqrt{3}}{8}, \sqrt{\frac{3}{32}}, \frac{\sqrt{3}}{4}, \cdots$.

30. The geometrical mean of two numbers is 4 and their sum is 10. Find the numbers.

31. The fourth term of a G.P. is 192, the seventh term is 12,288. Find the first term and the ratio.

32. If the same number be added to or subtracted from each term of a G.P., is the resulting series geometrical?

33. The product of the first and last of four numbers in G.P. is 64. Their quotient is also 64. Find the numbers.

34. The product of four numbers in G.P. is 81. The sum of the second and third terms is $\frac{1}{4}$. Find the numbers.

35. If every term of a G.P. be multiplied by the same number m, is the resulting series a G.P.? If so, what are the elements?

36. The sum of three numbers in G.P. is 42. The difference between the squares of the first and the second is 60. What are the numbers?

37. The difference between two numbers is 48. The arithmetical mean exceeds the geometrical mean by 18. Find the numbers.

38. Four numbers are in G.P. The difference between the first and the second is 4, the difference between the third and the fourth is 36. Find the numbers.

39. A ball falling from a height of 60 feet rebounds after each fall one third of the last descent. What distance has it passed over when it strikes the ground for the eighth time?

40. The difference between the first and the last of three terms in G.P. is four times the difference between the first and second terms. The sum of the numbers is 208. Find the numbers.

41. An invalid on a certain day was able to take a single step of 18 inches. If he was each day to walk twice as far as on the preceding day, how long before he can take a five-mile walk?

42. The difference between the first and the last of four numbers in G.P. is thirteen times the difference between the second and third terms. The product of the second and third terms is 3. Find the numbers.

137. Infinite series. When the number of terms of a G.P. is unlimited it is called an **infinite geometrical series.**

In the series $a, ar, ar^2, \cdots$, when $r > 1$, evidently each term is larger than the preceding term. The series is then called increasing. When $r < 1$, each term is smaller than the preceding term and the series is called decreasing.

Now in any case $s = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}.$

When $r > 1$, evidently r^n becomes very large for large values of n. For this case, then, the sum of the first n terms becomes very large for large values of n. In fact we can take enough terms so that s will exceed any number we may choose. If, however, $r < 1$, as n increases in value r^n becomes smaller and smaller. In fact we can choose n large enough so that r^n is as small as we wish, or as we say, approaches 0 as a limit. But since r^n may be made as small as we wish, ar^n also approaches 0 as a limit, and consequently $\frac{ar^n}{1-r}$ approaches 0 as a limit. Thus when $r < 1$ the value of the sum of the first n terms approaches $\frac{a}{1-r}$ as n becomes very great. This we express in other words by asserting that the sum of the infinite series

$$a + ar + ar^2 + \cdots, \text{ when } r < 1,$$

is

$$s_\infty = \frac{a}{1-r}.$$

EXERCISES

Find the sum of the following infinite series.

1. $6 + 3 + \frac{3}{2} + \cdots$.

Solution:

$$a = 6, \quad r = \tfrac{1}{2}.$$

$$s_\infty = \frac{a}{1-r} = \frac{6}{1-\frac{1}{2}} = \frac{6}{\frac{1}{2}} = 12.$$

2. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

3. $64 + 8 + 1 + \cdots$.

4. $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \cdots$.

5. $\frac{1}{5} + \frac{1}{15} + \frac{1}{45} + \cdots$.

6. $\frac{2}{3} + \frac{3}{2} + \frac{27}{8} + \cdots$.

7. $2 + .5 + .125 + \cdots$.

8. $\sqrt{2} + 1 + \frac{\sqrt{2}}{2}, \cdots$.

9. $(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + \cdots$.

10. How large a value of n must one take so that the sum of the first n terms of the following series differs from the sum to infinity by not more than .001?

(a) $8 + 4 + 2 + \cdots$.

Solution:

$$a = 8, \quad r = \tfrac{1}{2}.$$

$$s = \frac{a}{1-r} + \frac{ar^n}{1-r} = s_\infty + \frac{ar^n}{1-r}.$$

$$s - s_\infty = \frac{ar^n}{1-r}.$$

We must find for what value of n the expression $\frac{ar^n}{1-r}$ is less than .001.

$$\frac{8(\frac{1}{2})^n}{1-\frac{1}{2}}=\frac{8\cdot 2}{2^n}=\frac{16}{2^n}.$$

By trial we see that if $n=14$ the value of $\frac{16}{2^n}$ is $\frac{1}{1024}$, which is less than .001.

(b) $27+3+\frac{1}{3}+\cdots$.

(c) $4+\frac{1}{4}+\frac{1}{64}+\cdots$.

(d) $1+\frac{1}{15}+\frac{1}{225}+\cdots$.

(e) $64+16+4+\cdots$.

(f) $100+20+4+\cdots$.

(g) $60+20+6\frac{2}{3}+\cdots$.

11. What is the value of the following recurring decimal fractions?

(a) .212121 $\cdots$.

Solution: This decimal may be written in the form

$$\frac{21}{100}+\frac{21}{(100)^2}+\frac{21}{(100)^3}+\cdots.$$

Here $$a=\frac{21}{100},\ r=\frac{1}{100}.$$

$$s_\infty=\frac{a}{1-r}=\frac{.21}{1-.01}=\frac{.21}{.99}=\frac{7}{33}.$$

(b) .333 $\cdots$.

(c) .717171 $\cdots$.

(d) .801801 $\cdots$.

(e) .343343 $\cdots$.

(f) 1.43131 $\cdots$.

(g) 2.61414 $\cdots$.

ADVANCED ALGEBRA

CHAPTER XV

PERMUTATIONS AND COMBINATIONS

138. Introduction. Before dealing directly with the subject of the chapter we must answer the question, In how many distinct ways may two successive acts be performed if the first may be performed in p ways and the second may be performed in q ways? Suppose for example that I can leave a certain house by any one of four doors, and can enter another house by any one of five doors, in how many ways can I pass from one house to the other? If I leave the first house by a certain door, I have the choice of all five doors by which to enter the second house. Since, however, I might have left the first by any one of its four doors, there are $4 \cdot 5 = 20$ ways in which I may pass from one house to the other. This leads to the

THEOREM. *If a certain act may be performed in p ways, and if after this act is completed a second act may be performed in q ways, then the total number of ways in which the two acts may be performed is $p \cdot q$.*

With *each* of the p possible ways of performing the first act correspond q ways of performing the second act. Thus with *all* the p possible ways of performing the first act must correspond p times as many ways of performing the second act. That is, the two acts may be performed in $p \cdot q$ ways.

It is of course assumed in this theorem that the performance of the second act is entirely independent of the way in which the first act is performed.

EXERCISES

1. I have four coats and five hats. How many different combinations of coat and hat can I wear?

Solution: The first act consists in putting on one of my coats, which may be done in four ways; the second act consists in putting on one of my hats, which may be done in five ways. Thus I have $4 \cdot 5 = 20$ different combinations of coat and hat.

2. In how many ways may the two children of a family be assigned to five rooms if they each occupy a separate room?

3. A gentleman has four coats, six vests, and eight pairs of trousers. In how many different ways can he dress?

4. I can sail across a lake in any one of four sailboats and row back in any one of fifteen rowboats. In how many ways can I make the trip?

5. Two men wish to stop at a town where there are six hotels but do not wish quarters at the same hotel. In how many ways may they select hotels?

6. A man is to sail for England on a steamship line that runs ten boats on the route, and return on a line that runs only six. In how many different ways can he make the trip?

7. In walking from A to B one may follow any one of three roads; in going on from B to C one has a choice of five roads. In how many different ways can one walk from A to C?

139. Permutations. Each different arrangement either of all or of a part of a number of things is called a **permutation**.

Thus the digits 1, 2 have two possible permutations, taken both at a time, namely, 12 and 21.

The digits 1, 2, 3 have six different permutations when two are taken at a time, namely, 12, 13, 21, 23, 31, 32. For if we take 1 for the first place, we have a choice of 2 and 3 for the second place, and we get 12 and 13. If 2 is in the first place, we get 21 and 23. Similarly, we get 31 and 32. In this process it is noted that we can fill the first of the two places in any one of three ways; the second place can be filled in each case in only two ways. Thus by the Theorem, § 138, we should expect $3 \cdot 2 = 6$ permutations of three things taken two at a time. We observe that this product $3 \cdot 2$ has as its first factor 3, which is the total number of things considered. The number of factors is equal to the number of digits taken at a time, i.e. two. This leads to the general

THEOREM. *The number of permutations of n objects taken r at a time is*

$$n(n-1)\cdots(n-r+1). \tag{I}$$

This is symbolized by $P_{n,r}$.

This formula is easily remembered if one observes that the first factor is n, the total number of objects considered, and that the number of factors is r, the number of objects taken at a time. Thus $P_{7,3} = 7 \cdot 6 \cdot 5$.

We prove this theorem by complete induction.

First, let $r = 1$. There are evidently only n different arrangement of n objects, taking one object at a time, namely (assuming our objects to be the first n integers),

$$1, 2, 3, \cdots, n.$$

Let us take two objects at a time, i.e. let $r = 2$. Since there are n objects, we have n ways of filling the first of the two places. When that is filled there are $n - 1$ objects left, and any one may be used to fill the second place. Thus, by the Theorem, § 138, there are for $r = 2$

$$n(n-1)$$

different permutations.

Second, assume the form (I) for $r = m$,

$$P_{n,m} = n(n-1)\cdots(n-m+1). \tag{1}$$

We can fill the first m places in $P_{n,m}$ different ways since there are that number of permutations of n things taken m at a time. This constitutes the first act (§ 138). The second act consists in filling the $m + 1$st place, which may be done in $n - m$ ways by using any of the remaining $n - m$ objects. Thus the number of permutations of n things taken $m + 1$ at a time is

$$P_{n,m+1} = P_{n,m} \cdot (n-m) = n(n-1)(n-2)\cdots(n-m+1)(n-m),$$

which is the form that (1) assumes on replacing m by $m + 1$.

COROLLARY. *The number of permutations of n things taken all at a time is*

$$P_{n,n} = n(n-1)\cdots 2 \cdot 1 = n!.\text{*} \tag{2}$$

Taking $n = r$ in (I), we get (2).

* $n!$ is the symbol for $1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)n$, and is read *factorial n*.

EXERCISES

1. How many permutations may be formed from 8 letters taken four at a time?

Solution: $n = 8,\ r = 4,\ n - r + 1 = 5,$
$$P_{8,4} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

2. In how many different orders may 6 boys stand in a row?

3. How many different numbers less than 1000 can be formed from the digits 1, 2, 3, 4, 5 without repetition?

4. How many arrangements of the letters of the alphabet can be made taking three at a time?

5. How many numbers between 100 and 10,000 can be formed from the digits 1, 2, 3, 4, 5, 6 without repetition?

6. How many different permutations can be made of the letters in the word *compute* taking four at a time?

7. In a certain class there are 4 boys and 5 girls. In how many orders may they sit provided all the boys sit on one bench and all the girls on another?

HINT. Use Corollary § 139, and then Theorem, § 138.

8. I have 6 books with red binding and 3 with brown. In how many ways may I arrange them on a shelf so that all the books of one color are together?

140. Combinations. Any group of things that is independent of the order of the constituents of the group is called a **combination.**

The committee of men Jones, Smith, and Jackson is the same as the committee Jackson, Jones, and Smith. The sound made by striking simultaneously the keys EGC of a piano is the same as the sound made by striking CGE. In general a question involving the number of groups of objects that may be formed where the character of any group is unaltered by any change of order among its constituent parts is a question in combinations.

Suppose for example that we ask how many committees of three men can be selected from six men. If the men are called A, B, C, D, E, F, there are, by § 139, $6 \cdot 5 \cdot 4 = 120$ different arrangements or permutations of the six men in groups of three. But the permutations A, B, C; A, C, B; B, A, C, etc. ($3! = 6$ in all for the men A, B, and C), are all distinct, while evidently the six committees consisting of A, B, and C are identical. This is true for every distinct set of three men that we could select; that is, for the

six different permutations of any three men there is only one distinct committee. Hence the number of committees is one sixth the total number of permutations, or $\frac{P_{6,3}}{3!}$.

This leads to the general

THEOREM. *The number of combinations of n things taken r at a time is*

$$\frac{n(n-1)\cdots(n-r+1)}{r!}.$$

This is symbolized by $C_{n,r}$.

The number of permutations of n things taken r at a time is

$$P_{n,r} = n(n-1)\cdots(n-r+1).$$

In every group of r things which form *a single combination* there are (Cor., p. 145) $r!$ permutations. Thus there are $r!$ times as many permutations as combinations. That is,

$$C_{n,r} = \frac{P_{n,r}}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}. \qquad \text{(I)}$$

This formula is easily remembered if one observes that there is the same number of factors in the numerator as in the denominator. Thus

$$C_{10,3} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}.$$

COROLLARY. $C_{n,r} = C_{n,n-r}$.

Multiplying numerator and denominator of (I) by $(n-r)!$,

$$\begin{aligned} C_{n,r} &= \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1}{r!(n-r)!} \\ &= \frac{n!}{r!\,n-r!} \qquad \text{(II)} \\ &= \frac{n(n-1)\cdots(r+1)}{(n-r)!} \\ &= \frac{n(n-1)\cdots[n-(n-r)+1]}{(n-r)!} \\ &= C_{n,n-r}. \end{aligned}$$

This corollary saves computation in some cases. For instance, if we wish to compute $C_{19,17}$, it is more convenient to write $C_{19,17} = C_{19,2} = \frac{19\cdot 18}{1\cdot 2} = 171$ than the expression for $C_{19,17}$.

EXERCISES

1. How many committees of 5 men can be selected from a body of 10 men three of whom can serve as chairman but can serve in no other capacity?

Solution: There are 7 men who may fill 4 places on the committee.

$$C_{7,4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 35.$$

There are 3 men to select from for the remaining place of chairman, and the selection may be made in 3 ways. Thus the committee can be made up in $3 \cdot 35 = 105$ ways.

2. How many distinct crews of 8 men may be selected from a squad of 14 men?

3. How many distinct triangles can be drawn having their vertices in 10 given points no three of which are in a straight line?

4. How many distinct sounds may be produced on 9 keys of a piano by striking 4 at a time?

5. In how many ways can a crew of 8 men and a hockey team of 5 men be made up from 20 men?

6. In how many ways may the product $a \cdot b \cdot c \cdot d \cdot e \cdot f$ be broken up into factors each of which contains two letters?

7. If 8 points lie in a plane but no three in a straight line, how many straight lines can be drawn joining them in pairs?

8. How many straight lines can be drawn through n points taken in pairs no three of which are in the same straight line?

9. Seven boys are walking and approach a fork in the road. They agree that 4 shall turn to the right and the remainder turn to the left. In how many ways could they break up?

Solution: The number of groups of 4 boys that can be formed from the party is

$$C_{7,4} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4!} = 35.$$

For each group of 4 boys there remains only a single group of 3 boys. Thus the total number of ways in which the party can divide up is precisely 35.

10. If there are 12 points in space but no four in the same plane, how many distinct planes can be determined by the points?

HINT. Three points determine a plane.

11. Eight gentlemen meet at a party and each wishes to shake hands with all the rest. How many hand shakes are exchanged?

12. In how many ways can a baseball team of 9 men be selected from 14 men only two of whom can pitch but can play in no other position?

13. How many baseball teams can be selected from 15 men only four of whom can pitch or catch, provided these four can play in either of the two positions but cannot play elsewhere?

14. Two dormitories, one having 3 doors, the other having 5 doors, stand facing each other. A path runs from each door of one to every door of the other. How many paths are there?

15. Show that the number of ways in which $p + q$ things may be divided into groups of p and q things respectively is $\frac{(p+q)!}{p!\,q!}$.

16. Out of 8 consonants and 3 vowels how many words can be formed each containing 3 consonants and 2 vowels?

17. A boat's crew consists of 8 men, three of whom can row only on one side and two only on the other. In how many ways can the crew be arranged?

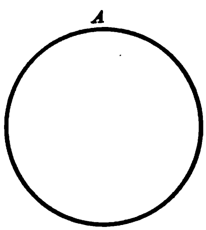
18. A pack of cards contains 52 distinct cards. In how many different ways can it be divided into 4 hands of 13 cards each?

19. Five points lie in a plane, but no three in any other plane. How many tetrahedrons can be formed with these points taken with two points not in the plane?

141. Circular permutations. By circular permutations we mean the various arrangements of a group of things around a circle.

THEOREM. *The number of orders in which n things may be arranged in a circle is $(n-1)!$.*

Suppose A is at the point at which we begin to arrange the digits $1, 2, 3, \cdots, n$. Suppose we start our arrangement of digits at A with a given digit a. We have then virtually $n-1$ places to fill by the remaining $n-1$ digits. Thus we get $(n-1)!$ (p. 145) permutations of the n digits *keeping a fixed.* But suppose we start our arrangement, that is, fill the place at A with any other digit, as b, and the remaining places in any order whatever. If we now go around the circle till we come to the digit a, the succession of digits from that point around the circle to a again must be one of the $(n-1)!$ orders



which we obtained when we took a as the initial figure. Thus the only *distinct* orders in which the n digits can be arranged on a circle are the $(n-1)!$ permutations we obtained by filling the first place with a.

EXERCISES

1. In how many orders can 6 men sit around a circular table?

Solution:

$$n = 6, \qquad n - 1 = 5, \qquad (n-1)! = 5! = 120.$$

2. In how many ways can 8 men sit around a circular table?

3. In how many ways may the letters of *live* be arranged on a circle?

4. In how many ways may the letters of *permutation* be arranged on a circle?

5. In how many ways can 4 men and 4 ladies sit around a table so that a lady is always between two men?

6. In how many ways may 4 men and their wives be seated around a table so that no man sits next his wife but the men and the women sit alternately?

7. In how many ways can six men and their wives be seated around a table so that each man sits between his wife and another lady?

8. In how many ways can 10 red flowers and 5 white ones be planted around a circular plot so that two and only two red ones are adjacent?

142. THEOREM. *The number of permutations of n things of which p are alike, taken all together, is* $\dfrac{n!}{p!}$.

If all the things were different, we should have $n!$ permutations. But since p of the n things are alike, any rearrangement of those p like things will not change the permutation. For any fixed arrangement of the n things there are $p!$ different arrangements of the p like things. Thus $\dfrac{1}{p!}$ of the $n!$ permutations are identical, and there are only $\dfrac{n!}{p!}$ distinct permutations of the n things p of which are alike.

COROLLARY. *If of n things p are of one kind, q of another kind, r of another, etc., then there are* $\dfrac{n!}{p!\,q!\,r!\ldots}$ *permutations of the n things taken all at a time.*

EXERCISES

1. How many distinct arrangements of the letters of the word *Cincinnati* are possible?

Solution: There are in all 10 letters, of which 3 are i, 2 are c, and 3 are n. Thus the number of arrangements is

$$\frac{10!}{3!\,3!\,2!} = \frac{\cancel{1}\cdot 2\cdot \cancel{3}\cdot \cancel{4}\cdot 5\cdot \cancel{6}\cdot 7\cdot 8\cdot 9\cdot 10}{\cancel{1}\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{1}\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{1}\cdot \cancel{2}}$$
$$= 2\cdot 5\cdot 7\cdot 8\cdot 9\cdot 10 = 50{,}400.$$

2. How many distinct arrangements of the letters of the word *parallel* can be formed?

3. How many signals can be made by hanging 15 flags on a staff if 2 flags are white, 3 black, 5 blue, and the rest red?

4. How many signals can be made by the flags in exercise 3 if a white one is at each extreme?

5. How many signals can be made by the flags in exercise 3 if a red flag is always at the top?

6. Would 3 dots, 2 dashes, and 1 pause be enough telegraphic symbols for the letters of the English alphabet, the numerals, and six punctuation marks?

CHAPTER XVI

COMPLEX NUMBERS

143. The imaginary unit. When we approached the solution of quadratic equations (p. 52) we saw that the equation $x^2 = 2$ was not solvable if we were at liberty to use only rational numbers, but that we must introduce an entirely new kind of number, defined as a sequence of rational numbers, if we wished to solve this equation. The excuse for introducing such numbers was not that we needed them as a means for more accurate measurement, — the rational numbers are entirely adequate for all mechanical purposes, — but that they are a *mathematical* necessity if we propose to solve equations of the type given.

A similar situation demands the introduction of still other numbers. If we seek the solution of

$$x^2 = -1, \tag{1}$$

we observe that there is no rational number whose square is -1. Neither can we define $\sqrt{-1}$ as a sequence of rational numbers which approach it as a limit. We may write the *symbol* $\sqrt{-1}$, but its meaning must be somewhat remote from that of $\sqrt{2}$, for in the latter case we have a process by which we can extract the square root and get a number whose square is as nearly equal to 2 as we desire. This is not possible in the case of $\sqrt{-1}$. In fact this symbol differs from 1 or any real number not merely in degree but in kind. One cannot say $\sqrt{-1}$ is greater or less than a real number, any more than one can compare the magnitude of a quart and an inch.

$\sqrt{-1}$ is symbolized by i and is called the **imaginary unit.** The term "imaginary" is perhaps too firmly established in mathematical literature to warrant its discontinuance. It should be kept in mind, however, that it is really no more and no less

imaginary than the negative numbers or the irrational numbers are. So far as we have yet gone it is merely a thing that satisfies equation (1). When, however, we have defined the various operations on it and ascribed to it the various characteristic properties of numbers we shall be justified in calling it a number.

Just as we built up from the unit 1 a system of real numbers, so we build up from $\sqrt{-1} = i$ a system of imaginary numbers. The fact that we cannot measure $\sqrt{-1}$ on a rule should cause no more confusion than our inability exactly to measure $\sqrt[3]{2}$ on a rule. Just as we were able to deal with irrational numbers as readily as with integers when we had defined what we meant by the four operations on them, so will the imaginaries become indeed numbers with which we can work when we have defined the corresponding operations on them.

144. Addition and subtraction of imaginary numbers. We write

$$\begin{aligned} 0 &= 0\,i, \\ i + i &= 2\,i, \\ i + i + i &= 3\,i, \\ \underbrace{i + i + \cdots + i}_{n \text{ terms}} &= ni. \end{aligned} \tag{I}$$

Also just as we pass from a rational to an irrational multiple of unity by sequences, so we pass from a rational to an irrational multiple of the imaginary unit. Thus we write $a\sqrt{-1}$, or ai, where a represents any real number. Consistently with § 76 we write

$$\pm\sqrt{-a^2} = \pm\sqrt{a^2\cdot(-1)} = \pm\sqrt{a^2}\cdot\sqrt{-1} = \pm\, a\sqrt{-1} = \pm\, ai. \tag{II}$$

We speak of a positive or a negative imaginary according as the radical sign is preceded by a positive or a negative sign.

We also define addition and subtraction of imaginaries as follows:

$$ai \pm bi = (a \pm b)\,i, \tag{III}$$

where a and b are any real numbers.

ASSUMPTION. *The commutative and associative laws of multiplication and addition of real numbers, § 10, we assume to hold for imaginary numbers.*

145. Multiplication and division of imaginaries. We have already virtually defined the multiplication of imaginaries by real numbers by formula (I). Consistently with § 76 we define

$$\sqrt{-1}\cdot\sqrt{-1}=i\,i=i^2=-1.$$

Thus $\sqrt{-a}\cdot\sqrt{-b}=\sqrt{a}\cdot\sqrt{b}\,i\,i=\sqrt{ab}\cdot(-1)=-\sqrt{ab}.$

The law of signs in multiplication may be expressed verbally as follows:

The product of imaginaries with like signs before the radical is a negative real number. The product of imaginaries with unlike signs is a positive real number.

For instance, $-\sqrt{-4}\cdot\sqrt{-9}=-2\cdot 3\cdot i^2=6.$

We also note that

$$i^2=-1,\ i^3=-i,\ i^4=1,\ i^5=i,\ \cdots.$$

And, in general, $i^{4n+k}=i^k,\ k=0,\ 1,\ 2,\ 3.$

We define division of imaginaries as follows:

$$\sqrt{-a}\div\sqrt{-b}=\frac{\sqrt{a}\cdot i}{\sqrt{b}\cdot i}=\sqrt{\frac{a}{b}}.$$

In operating with imaginary numbers, a number of the form $\sqrt{-a}$ should always be written in the form $\sqrt{a}\,i$ before performing the operation. This avoids temptation to the following error:

$$\sqrt{-a}\cdot\sqrt{-b}=\sqrt{(-a)\cdot(-b)}=\sqrt{ab}.$$

EXERCISES

Simplify the following:

1. $\sqrt{-8}\cdot\sqrt{-2}.$

Solution: $\sqrt{-8}\cdot\sqrt{-2}=\sqrt{8}\cdot i\cdot\sqrt{2}\cdot i=\sqrt{2\cdot 8}\cdot i^2=4\cdot(-1)=-4.$

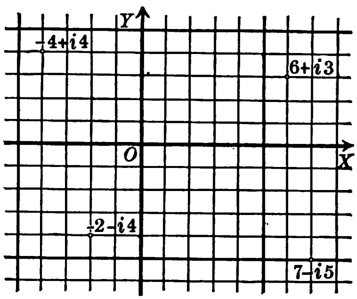
2. $\dfrac{1}{i^5}.$

Solution: $\dfrac{1}{i^5}=\dfrac{i^3}{i^8}=\dfrac{i^3}{(i^4)^2}=\dfrac{-i}{1}=-i.$

3. i^{17}. **4.** i^{24}. **5.** i^{18}.
6. $\sqrt{-36}$. **7.** $\sqrt{-64}$. **8.** $2i \cdot 3i$.
9. $\sqrt{-x^{2n}}$. **10.** $\sqrt{-3x^2a^4}$. **11.** $\sqrt{-x^2}$.
12. $\sqrt{2}\sqrt{-8}$. **13.** $\sqrt{-2}\sqrt{-6}$. **14.** $\sqrt{-3}\sqrt{-27}$.
15. $\frac{1}{i^3}$. **16.** $\frac{1}{i}$. **17.** $\frac{\sqrt{-4}}{\sqrt{-2}}$.
18. $\frac{\sqrt{-a}}{\sqrt{-b}}$. **19.** $\sqrt{-i^2}$. **20.** $\sqrt{-i^4}$.

146. Complex numbers. The solution of the quadratic equation with negative discriminant (p. 71) affords us an expression which consists of a real number connected with an imaginary number by a $+$ or $-$ sign. Such an expression is called a **complex number.** It consists of two parts which are of different kinds, the real part and the imaginary part. Thus $6 + 4i$ means 6 1's $+$ 4 i's. Obviously, to any pair of real numbers (x, y) corresponds a complex number $x + iy$, and conversely.

147. Graphical representation of complex numbers. We have represented all real numbers on a single straight line. When we wished to represent two numbers simultaneously, we made use of the plane, and assumed a one-to-one correspondence between the points on the plane and the pairs of numbers (x, y). The general complex number $x + iy$ depends on the values of the independent real numbers x and y, and may then properly be represented by a point on a plane. We represent real numbers on the X axis, imaginary numbers on the Y axis, and the complex number $x + iy$ by the point (x, y) on the plane. Thus the complex numbers $6 + i3$, $-4 + i4$, $7 - i5$, $-2 - i4$ are represented by points on the plane as indicated in the figure.



148. Equality of complex numbers. We define the two complex numbers $a + ib$ and $c + id$ to be equal when and only when $a = c$ and $b = d$.

Symbolically $$a + ib = c + id$$

when and only when $$a = c,\ b = d.$$

The definition seems reasonable, since 1 and i are different in kind, and we should not expect any real multiple of one to cancel any real multiple of the other.

Similarly, if we took not abstract expressions as 1 and i for units but concrete objects as trees and streets, we should say that

$$a \textit{ trees} + b \textit{ streets} = c \textit{ trees} + d \textit{ streets}$$

when and only when $a = c$ and $b = d$.

PRINCIPLE. *When two numerical expressions involving imaginaries are equal to each other, we may equate real parts and imaginary parts separately.*

The graphical interpretation of the definition of equality is that equal complex numbers are always represented by the same point on the plane.

From the definition given we see that $a + ib = 0$ when and only when $a = b = 0$.

ASSUMPTION. *We assume that complex numbers obey the commutative and associative laws and the distributive law given in* § 10. *We also assume the same rules for parentheses as given in* § 15.

This assumption enables us to define the fundamental operations on complex numbers.

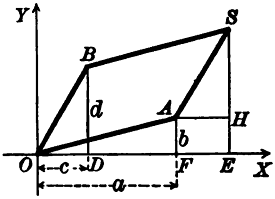
149. Addition and subtraction. By applying the assumptions just made we obtain the following symbolical expression for the operations of addition and subtraction of any two complex numbers $a + ib$ and $c + id$:

$$a + ib \pm (c + id) = a \pm c + i(b \pm d).$$

RULE. *To add (subtract) complex numbers, add (subtract) the real and imaginary parts separately.*

150. Graphical representation of addition. We now proceed to give the graphical interpretation of the operations of addition and subtraction.

THEOREM. *The sum of two numbers $A = a + ib$ and $B = c + id$ is represented by the fourth vertex of the parallelogram formed on OA and OB as sides.*



Let $OASB$ be a parallelogram. Draw $ES \perp OE$, $AH \perp ES$, $BD\ (= d) \perp OE$. $\triangle AHS = \triangle ODB$ since their sides are parallel, and $OB = AS$.

Thus $\qquad DB = HS = d,$

$\qquad OD = AH = c.$

Thus $\qquad ES = EH + HS = b + d,$

$\qquad OE = OF + FE = a + c,$

and S has coördinates $(a + c, b + d)$ and represents the sum of A and B, by § 149.

EXERCISES

1. The difference $A - B$ of two numbers $A = a + ib$ and $B = c + id$ is represented by the extremity D of the line OD drawn from the origin parallel to the diagonal BA of the parallelogram formed on OB and OA as sides.

2. Represent graphically the following expressions.

(a) $1 + i$.	(b) $-4 - 2i$.
(c) $6 - i$.	(d) $-8 + 4i$.
(e) $2 + 4i$.	(f) $(1 + i) + (2 + i)$.
(g) $(2 - i) - (6 - 3i)$.	(h) $(1 - i) - (1 - 2i)$.
(i) $(2 + 4i) - (1 - 3i)$.	(j) $4(1 + i) - 2(2 - 3i)$.
(k) $(6 - 2i) + (2 + 3i)$.	(l) $(5 + 3i) + (-1 - 6i)$.

151. Multiplication of complex numbers. The assumption of § 148 enables us to multiply complex numbers by the following

RULE. *To multiply the complex number $a + ib$ by $c + id$, proceed as if they were real binomials, keeping in mind the laws for multiplying imaginaries.*

Thus

$$\begin{array}{r} a + ib \\ c + id \\ \hline ac + icb + iad + (i)^2 bd = ac - bd + i(cb + ad). \end{array}$$

152. Conjugate complex numbers. Complex numbers that differ only in the sign of their imaginary parts are called **conjugate complex numbers**, or **conjugate imaginaries.**

THEOREM. *The sum and the product of conjugate complex numbers are real numbers.*

Thus
$$a + ib + a - ib = 2a,$$
$$(a + ib)(a - ib) = a^2 + b^2.$$

153. Division of complex numbers. The quotient of two complex numbers may now be expressed as a single complex number.

RULE. *To express the quotient* $\dfrac{a+ib}{c+id}$ *in the form* $x + iy$, *rationalize the denominator, using as a rationalizing factor the conjugate of the denominator.*

Thus
$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id}$$
$$= \frac{ac + bd - i(ad - bc)}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} - i\frac{ad - bc}{c^2 + d^2}. \tag{1}$$

We have now defined the fundamental operations on complex numbers and shall make frequent use of them. If the question remains in one's mind, "After all, what are they?" the answer is this: They are things for which we have defined the fundamental operations of numbers and, since they have the properties of numbers, must be called numbers, just as a flower that has all the characteristic properties of a known species is thereby determined to belong to that species. Furthermore, our operations have been so defined that if the imaginary parts of the complex numbers vanish and the numbers become real, the expression defining any operation on complex numbers reduces to one defining the same operation on the real part of the number. Thus in (1) above, if $b = d = 0$, the expression reduces to
$$\frac{a}{c} = \frac{a}{c}.$$

EXERCISES

Carry out the indicated operations.

1. $(2+\sqrt{-2})(4+\sqrt{-5})$.

Solution:
$$\begin{aligned} 2+\sqrt{-2} &= 2+\sqrt{2(-1)} = 2+i\sqrt{2} \\ 4+\sqrt{-5} &= 4+\sqrt{5(-1)} = \underline{4+i\sqrt{5}} \\ & \qquad\qquad 8-\sqrt{10}+i4\sqrt{2}+i2\sqrt{5} \end{aligned}$$

2. $5 \div \sqrt{2} - i\sqrt{3}$.

Solution:
$$\frac{5}{\sqrt{2}-i\sqrt{3}} = \frac{5(\sqrt{2}+i\sqrt{3})}{(\sqrt{2}-i\sqrt{3})(\sqrt{2}+i\sqrt{3})} = \frac{5\sqrt{2}+i5\sqrt{3}}{2+3} = \sqrt{2}+i\sqrt{3}.$$

3. $(1+i)^4$.

4. $(1+i)^8$.

HINT. Develop by the binomial theorem.

5. $(a+ib)^4$.

6. $(\sqrt{a}+\sqrt{-a})^2$.

7. $(x+iy)^2$.

8. $(x+iy)^2+(x-iy)^2$.

9. $\sqrt{1+i}\cdot\sqrt{1-i}$.

10. $(\sqrt{3}+i\sqrt{2})(\sqrt{2}+i\sqrt{3})$.

11. $(\sqrt{1+i}+\sqrt{1-i})^2$.

12. $(a\sqrt{b}+ic\sqrt{d})(a\sqrt{b}-ic\sqrt{d})$.

13. $(\sqrt{a}+i\sqrt{b})(\sqrt{a}-i\sqrt{b})$.

14. $(2\sqrt{7}+i3\sqrt{8})(3\sqrt{7}-i10\sqrt{2})$.

15. $\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$.

16. $\dfrac{1+i}{1-i}$.

17. $\dfrac{3}{\sqrt{2}+\sqrt{-1}}$.

18. $\dfrac{4}{1+\sqrt{-3}}$.

19. $\dfrac{(1-i)^2}{(1+i)^3}$.

20. $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^3$.

21. $\left(\dfrac{\sqrt{3}+i}{2}\right)^6$.

22. $\left(\dfrac{1+i}{\sqrt{2}}\right)^4$.

23. $\dfrac{a+i\sqrt{1-a^2}}{a-i\sqrt{1-a^2}}$.

24. $\dfrac{5}{\sqrt{2}-i\sqrt{3}}$.

25. $\dfrac{21}{4+3i\sqrt{6}}$.

26. $\dfrac{\sqrt{-a}+\sqrt{-b}}{\sqrt{-a}-\sqrt{-b}}$.

27. $\dfrac{\sqrt{3}+i\sqrt{2}}{\sqrt{3}-i\sqrt{2}}$.

28. $\left(\dfrac{1+i\sqrt{3}}{2}\right)^3$.

29. $\dfrac{a+ib}{a-ib}+\dfrac{c+id}{c-id}$.

30. $\dfrac{29}{4+7\sqrt{-5}}$.

31. $\dfrac{64}{1+3\sqrt{-7}}$.

32. $\dfrac{1}{(1+i)^2}+\dfrac{1}{(1-i)^2}$.

33. $\dfrac{\sqrt{1+a}+i\sqrt{1-a}}{\sqrt{1+a}-i\sqrt{1-a}} - \dfrac{\sqrt{1-a}+i\sqrt{1+a}}{\sqrt{1-a}-i\sqrt{1+a}}$.

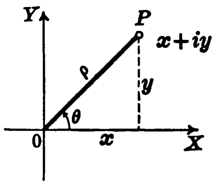
34. Find three roots of the equation $x^3 - 1 = 0$ and represent the roots as points on the plane.

35. Find four roots of the equation $x^4 - 1 = 0$ and represent the roots as points on the plane.

36. Find six roots of $x^6 - 1 = 0$ and represent the roots as points on the plane. Show graphically that the sum of the six roots is zero.

37. Find three roots of $x^3 - 8 = 0$ and represent the roots as points on the plane. Show graphically that the sum of the three roots is zero.

154. Polar representation. The graphical representation of complex numbers given in § 147 gives a simple graphical interpretation of the operations of addition and subtraction, but the graphical meaning of the operations of multiplication and division may be given more clearly in another manner. We have seen that we may represent $x + iy$ by the point $P(x, y)$ on the plane. Represent the angle between OP and the X axis by θ. This angle is called the **argument** of the complex number $x + iy$. Represent the line OP by ρ. This is called the **modulus** of $x + iy$. Then from the figure



$$x = \rho \cos \theta, \tag{1}$$

$$y = \rho \sin \theta, \tag{2}$$

$$x^2 + y^2 = \rho^2. \tag{3}$$

Hence the complex number $x + iy$ may be written in the form

$$x + iy = \rho(\cos \theta + i \sin \theta), \tag{4}$$

when the relations between x, y and ρ, θ are given by (1), (2), and (3). A number expressed in this way is in **polar form**, and may be designated by (ρ, θ). We observe that a complex number lies on a circle whose center is the origin and whose radius is the modulus of the number. The argument is the angle between the axis of real numbers and the line representing the modulus.

155. Multiplication in polar form. If we have two numbers $\rho(\cos \theta + i \sin \theta)$ and $\rho'(\cos \theta' + i \sin \theta')$, we may multiply them and obtain

$$\rho(\cos\theta + i\sin\theta)\rho'(\cos\theta' + i\sin\theta')$$
$$= \rho\rho'[(\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')]$$

By the addition theorem in Trigonometry
$$= \rho\rho'[\cos(\theta+\theta') + i\sin(\theta+\theta')] \quad (1)$$
$$= R(\cos\Theta + i\sin\Theta). \quad (2)$$

In this product $\rho\rho'$ is the new modulus and $\theta+\theta'$ the new argument. We may now make the following statement: *The product of the two numbers $\rho(\cos\theta + i\sin\theta)$ and $\rho'(\cos\theta' + i\sin\theta')$ has as its modulus $\rho\rho'$ and as its argument $\theta+\theta'$.* Thus the product of two numbers is represented on a circle whose radius is the product of the radii of the circles on which the factors are represented. The argument of the product is the sum of the arguments of the factors.

156. Powers of numbers in polar form. When the two factors of the preceding section (ρ, θ) and (ρ', θ') are equal, that is, when $\rho = \rho'$ and $\theta = \theta'$, the expression (1) assumes the form

$$[\rho(\cos\theta + i\sin\theta)]^2 = \rho^2(\cos 2\theta + i\sin 2\theta). \quad (1)$$

This suggests as a form for the nth power of a complex number

$$[\rho(\cos\theta + i\sin\theta)]^n = \rho^n(\cos n\theta + i\sin n\theta). \quad (2)$$

The student should establish this expression by the method of complete induction. The theorem expressed by (2) is known as **DeMoivre's** theorem. Stated verbally it is as follows: The modulus of the nth power of a number is the nth power of its modulus. The argument of the nth power of a number is n times its argument.

EXERCISES

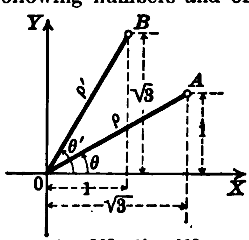
Plot, find the arguments and moduli of the following numbers and of their products.

1. $1 + i\sqrt{3}$, $\sqrt{3} + i$.

Solution:

Let $\sqrt{3} + i = \rho(\cos\theta + i\sin\theta)$,
$1 + i\sqrt{3} = \rho'(\cos\theta' + i\sin\theta')$.

Then by (1), (2), (3), § 154, $\rho = 2$; $\rho' = 2$.
$1 = 2\cos\theta$, hence $\theta = 60°$;
$1 = 2\sin\theta'$, hence $\theta' = 30°$.



$\theta = 30°$, $\theta' = 60°$

Thus if the product has the form $R(\cos\Theta + i\sin\Theta)$, we have by § 155, $R = \rho\rho' = 4$, $\Theta = \theta + \theta' = 90°$.

2. $1+i,\ 2+i.$

3. $(1-i)^3.$

4. $3+3i,\ 2-i\sqrt{12}.$

5. $2i,\ 1-i\sqrt{3}.$

6. $\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2.$

7. $-\frac{1}{3}+\frac{i}{3},\ -2-2i.$

8. $4i,\ -\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}.$

9. $\frac{1}{2}+\frac{i\sqrt{3}}{2},\ \frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}.$

10. $[2(\cos 15° + i\sin 15°)]^3.$

11. $[\frac{1}{2}(\cos 30° + i\sin 30°)]^4.$

12. $[\frac{3}{2}(\cos 120° + i\sin 120°)]^2.$

13. $[2(\cos 135° + i\sin 135°)]^4.$

14. $[\frac{4}{3}(\cos 180° + i\sin 180°)]^3.$

15. $[\frac{4}{3}(\cos 315° + i\sin 315°)]^2.$

157. Division in polar form. If we have, as before, two complex numbers in polar form (ρ, θ) and (ρ', θ'), we may obtain their quotient as follows.

$$\frac{\rho(\cos\theta + i\sin\theta)}{\rho'(\cos\theta' + i\sin\theta')}$$

Rationalizing,
$$= \frac{\rho\rho'(\cos\theta + i\sin\theta)(\cos\theta' - i\sin\theta')}{\rho'^2(\cos\theta' + i\sin\theta')(\cos\theta' - i\sin\theta')}$$

§ 152 and § 153,
$$= \frac{\rho\rho'[\cos(\theta-\theta') + i\sin(\theta-\theta')]}{\rho'^2(\cos^2\theta' + \sin^2\theta')}$$

Since $\sin^2\theta + \cos^2\theta = 1$,
$$= \frac{\rho}{\rho'}[\cos(\theta-\theta') + i\sin(\theta-\theta')]$$
$$= R(\cos\Theta + i\sin\Theta).$$

We may now make the following statement: *The quotient of two complex numbers has as its modulus the quotient of the moduli of the factors, and as its argument the difference of the arguments of the factors.*

158. Roots of complex numbers. We have seen that the square of a number has as its modulus the square of the original modulus, while the argument is twice the original argument.

This would suggest that the square root of any number, as (ρ, θ), would have $\sqrt{\rho}$ as its modulus and $\frac{\theta}{2}$ as its argument. Since every real number has two square roots, we should expect the same fact to hold here. Consider the two numbers

$$\sqrt{\rho}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \text{ and } \sqrt{\rho}\left[\cos\left(\frac{\theta}{2} + 180°\right) + i\sin\left(\frac{\theta}{2} + 180°\right)\right],$$

where $\sqrt{\rho}$ is the principal square root of ρ (§ 72). The square of the first is (ρ, θ), by § 155. That the square of the second is the same is evident if we keep in mind the fact that

$$\cos(\theta + 360°) = \cos\theta$$

and

$$\sin(\theta + 360°) = \sin\theta.$$

Thus $\sqrt{\rho(\cos\theta + i\sin\theta)}$

$$= \begin{cases} \sqrt{\rho}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right) \text{ or} \\ \sqrt{\rho}\left[\cos\left(\dfrac{\theta}{2} + 180°\right) + i\sin\left(\dfrac{\theta}{2} + 180°\right)\right]. \end{cases}$$

The graphs of these two numbers are situated at points symmetrical to each other with respect to the origin.

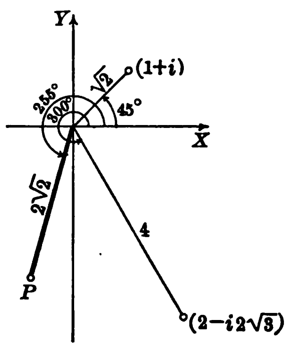
We may obtain as the corresponding expression for the higher roots of complex numbers the following:

$$\sqrt[n]{\rho(\cos\theta + i\sin\theta)} = \sqrt[n]{\rho}\left[\cos\left(\frac{\theta + k\,360°}{n}\right) + i\sin\left(\frac{\theta + k\,360°}{n}\right)\right],$$

where for a given value of n, k takes on the values $0, 1, \cdots, n-1$, and where $\sqrt[n]{\rho}$ indicates the real positive nth root of ρ.

EXERCISES

Perform the indicated operations and plot:



1. $2 - 2\sqrt{3}\,i \div 1 + i$.

Solution:

Let $\quad 1 + i = \rho(\cos\theta + i\sin\theta)$,

$\quad 2 - 2\sqrt{3}\,i = \rho'(\cos\theta' + i\sin\theta')$.

Then $\quad \rho = \sqrt{1^2 + 1^2} = \sqrt{2}$,

$\quad \rho' = \sqrt{2^2 + (-2\sqrt{3})^2} = 4$.

By (1) and (2), § 154,

$$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}, \text{ hence } \theta = 45°.$$

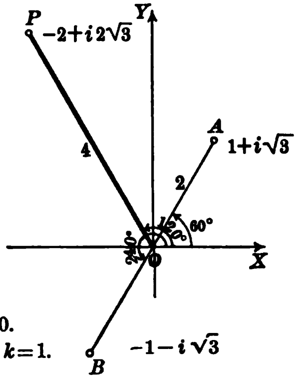
Similarly, $\sin\theta' = -\dfrac{2\sqrt{3}}{4} = -\dfrac{\sqrt{3}}{2}$,

$\cos\theta' = \frac{2}{4} = \frac{1}{2}$, hence $\theta' = 300°$.

Thus $\dfrac{2-2\sqrt{3}\,i}{1+i} = R(\cos\Theta + i\sin\Theta) = \dfrac{4(\cos 300° + i\sin 300°)}{\sqrt{2}(\cos 45° + i\sin 45°)}.$

Hence by § 157, $R = \dfrac{4}{2\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2},\ \Theta = 300° - 45° = 255°.$

2. $\sqrt{-2+2\sqrt{3}\,i}.$



Let $-2+2\sqrt{3}\,i = \rho(\cos\theta + i\sin\theta)$.

Then (§ 154) $\rho = 4,\ \cos\theta = -\frac{2}{4} = -\frac{1}{2}$,

and $\theta = 120°$.

$\sqrt{-2+2\sqrt{3}\,i} = \sqrt{4(\cos 120° + i\sin 120°)}$

By § 158, $= \sqrt{4}\left[\cos\left(\dfrac{120° + k\,360°}{2}\right) + i\sin\left(\dfrac{120° + k\,360°}{2}\right)\right]$

(where $k = 0$ or 1)

$= 2(\cos 60° + i\sin 60°) = 1 + i\sqrt{3}$, when $k = 0$.

$= 2(\cos 240° + i\sin 240°) = -1 - i\sqrt{3}$, when $k = 1$.

3. $\sqrt{\sqrt{2} + i\sqrt{2}}.$

4. $\sqrt{-1-i}.$

5. $-1 + i \div 1 - i.$

6. $\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2} \div 1 + i.$

7. $1 + i \div \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i.$

8. $-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2} \div \dfrac{1}{4} + \dfrac{i}{4}.$

9. $2 - i\sqrt{12} \div 3 + 3i.$

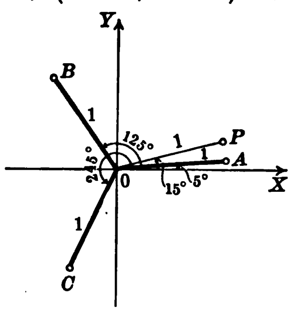
10. $-2\sqrt{2} - 2\sqrt{2}\,i \div -2 + 2\sqrt{3}\,i.$

11. $\sqrt[3]{1(\cos 15° + i\sin 15°)}.$

Solution:

$\sqrt[3]{1(\cos 15° + i\sin 15°)} = \sqrt[3]{1}\left[\cos\left(\dfrac{15° + k\cdot 360°}{3}\right) + i\sin\left(\dfrac{15° + k\cdot 360°}{3}\right)\right]$

(where $k = 0$, 1, or 2).



$$= \begin{cases} 1(\cos 5° + i\sin 5°), \text{ when } k = 0, \\ 1(\cos 125° + i\sin 125°), \text{ when } k = 1, \\ 1(\cos 245° + i\sin 245°), \text{ when } k = 2. \end{cases}$$

12. $\sqrt[3]{i}.$

13. $\sqrt[4]{16\,i}.$

14. $\sqrt[3]{2 + 2\sqrt{3}\,i}.$

15. $\sqrt[3]{\cos 330° + i\sin 330°}.$

16. $\sqrt[3]{27(\cos 75° + i \sin 75°)}$. **17.** $\sqrt[4]{16(\cos 200° + i \sin 200°)}$.

18. Solve the following equations and plot their roots.

(*a*) $x^5 - 1 = 0$.

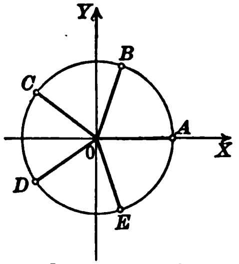
Solution: $x^5 = 1$, or $x = \sqrt[5]{1}$.

Let $1 = 1 + 0 \cdot i = \rho(\cos\theta + i \sin\theta)$. Then $\rho = 1$, $\theta = 0°$.

$$x = \sqrt[5]{1(\cos 0° + i \sin 0°)} = \sqrt[5]{1}\left[\cos\left(\frac{0° + k \cdot 360°}{5}\right) + i \sin\left(\frac{0° + k \cdot 360°}{5}\right)\right]$$

(where k takes on the values 0, 1, 2, 3, 4)

$$= \begin{cases} \cos 0° + i \sin 0° = 1, \text{ when } k = 0, \\ \cos 72° + i \sin 72°, \text{ when } k = 1, \\ \cos 144° + i \sin 144°, \text{ when } k = 2, \\ \cos 216° + i \sin 216°, \text{ when } k = 3, \\ \cos 288° + i \sin 288°, \text{ when } k = 4. \end{cases}$$



These numbers we observe lie on a circle of unit radius at the vertices of a regular pentagon.

(b) $x^4 - 1 = 0$. (c) $x^3 - 1 = 0$. (d) $x^5 - 32 = 0$.

(e) $x^6 - 1 = 0$. (f) $x^8 - 1 = 0$. (g) $x^3 - 27 = 0$.

CHAPTER XVII

THEORY OF EQUATIONS

159. Equation of the nth degree. Any equation in one variable in which the coefficients are rational numbers can be put in the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \tag{1}$$

where a_0 is positive and $a_0, \cdots, a_n$ are all integers.

The symbol $f(x)$ is read "f of x" and is merely an abbreviation for the right-hand member of the equation. Often we wish to replace x in the equation by some constant, as α, -2, or 0. We may symbolize the result of this substitution by $f(\alpha)$, $f(-2)$, or $f(0)$.

Thus $$f(b) = a_0b_n + a_1b^{n-1} + \cdots + a_n.$$

We symbolize other expressions similarly by $\phi(x)$, $Q(x)$, etc.

When we speak of an equation we assume that it is in the form of (1). This equation is also written in the form

$$x^n + b_1x^{n-1} + \cdots b_n = 0, \tag{2}$$

where $$b_1 = \frac{a_1}{a_0},\ b_2 = \frac{a_2}{a_0},\ \cdots b_n = \frac{a_n}{a_0}.$$

The b's are integers only when $a_1, a_2, \cdots, a_n$ are multiples of a_0.

160. Remainder theorem. We now prove the following important fact.

THEOREM. *When $f(x)$ is divided by $x - c$, the remainder is $f(x)$ with c substituted in place of the variable.*

Divide the equation (1) by $x - c$. Let R be the remainder, which must (§ 26) be of lower degree in x than the divisor; that is, in this case, since $x - c$ is the divisor, R must be a constant and not involve x at all. Let the quotient, which is of degree $n - 1$ in x, be represented by $Q(x)$.

Then $$\frac{f(x)}{x-c} \equiv Q(x) + \frac{R}{x-c}.$$

Clearing of fractions,

$$f(x) \equiv Q(x)(x-c) + R.$$

But since this equation is an *identity* it is *always satisfied* whatever numerical value x may have (§ 53).

Let $x = c$.

Then $f(c) = a_0c^n + a_1c^{n-1} + \cdots + a_n = Q(c)(c-c) + R.$

But since $c - c = 0$, $Q(c)(c-c) = 0$, and

$$R = a_0c^n + a_1c^{n-1} + \cdots + a_n = f(c).$$

COROLLARY. *If c is a root of $f(x) = 0$, then $x - c$ is a factor of the left-hand member.*

For if c is a root of the left-hand member, it satisfies that member and reduces it to zero when substituted for x. Thus by the previous theorem we have, since

$$a_0c^n + a_1c^{n-1} + \cdots a_n = R = 0,$$

$$f(x) = Q(x)(x-c).$$

161. Synthetic division. In order to plot by the method of § 103 the equation

$$y = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

when the a's are replaced by integers, we should be obliged laboriously to substitute for x successive integers and find corresponding values of y, which for large values of n involves considerable computation. We can make use of the preceding theorem to lighten this labor. The object is to find, with the least possible computation, the remainder when the polynomial $f(x)$ is divided by a factor of form $x - c$, which by the preceding theorem is the value of $f(x)$ when x is replaced by c, that is, the value of y corresponding to $x = c$. For illustration, let

$$f(x) = 2x^4 - 3x^3 + x^2 - x - 9 \text{ and } c = 2.$$

By long division we have

$$
\begin{array}{rrrrrrl}
\underline{x-2|} & \mathbf{2}x^4 & -\,3x^3 & +\;x^2 & -\;x & -\;9 & \underline{|\,2x^3+x^2+3x+5} \\
& \underline{2x^4} & \underline{-\,4x^3} & & & & \\
& & \mathbf{1}x^3 & +\;x^2 & & & \\
& & \underline{1x^3} & \underline{-\,2x^2} & & & \\
& & & \mathbf{3}x^2 & -\;x & & \\
& & & \underline{3x^2} & \underline{-\,6x} & & \\
& & & & \mathbf{5}x & -\;9 & \\
& & & & \underline{5x} & \underline{-\,10} & \\
& & & & & +\;1 &
\end{array}
$$

We can abbreviate this process by observing the following facts. Since x is here only the carrier of the coefficient, we may omit writing it. Also we need not rewrite the first number of the partial product, as it is only a repetition of the number directly above it in full-faced type. Our process now assumes the form

$$
\begin{array}{rrrrrrl}
\underline{1-2|} & \mathbf{2} & -\,3 & +\,1 & -\,1 & -\;9 & \underline{|\,2+1+3+5} \\
& & \underline{-\,4} & & & & \\
& & +\,\mathbf{1} & & & & \\
& & & \underline{-\,2} & & & \\
& & & +\,\mathbf{3} & & & \\
& & & & \underline{-\,6} & & \\
& & & & +\,\mathbf{5} & & \\
& & & & & \underline{-\,10} & \\
& & & & & +\;\mathbf{1} &
\end{array}
$$

Since the minus sign of the 2 changes every sign in the partial product, if we replace -2 by $+2$ we may *add* the partial product to the number in the dividend instead of subtracting. This is also desirable since the number which we are substituting for x is 2, not -2. Thus, bringing all our figures on one line and placing the number substituted for x at the right hand, we have

$$
\begin{array}{rrrrrl}
2 & -\,3 & +\,1 & -\,1 & -\;9 & \underline{|2} \\
& +\,4 & +\,2 & +\,6 & +\,10 & \\ \hline
2 & +\,1 & +\,3 & +\,5 & +\;1 &
\end{array}
$$

We observe that the figures in the lower line, 2, 1, 3, 5, up to the remainder are the coefficients of the quotient $2x^3 + x^2 + 3x + 5$.

RULE FOR SYNTHETIC DIVISION. *Write the coefficients of the polynomial in order, supplying 0 when a coefficient is lacking.*

Multiply the number to be substituted for x by the first coefficient, and add (algebraically) the product to the next coefficient.

Multiply this sum by the number to be substituted for x, add to the next coefficient, and proceed until all the coefficients are used. The last sum obtained is the remainder and also the value of the polynomial when the number is substituted for the variable.

162. Proof of the rule for synthetic division. This rule we now prove in general by complete induction. Let the polynomial be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n.$$

Let the number to be substituted for x be α.

First. Let $n = 2$. Carry out the rule on $a_0x^2 + a_1x + a_2$.

We have

$$\begin{array}{llll} a_0 & +a_1 & +a_2 & \underline{|\alpha} \\ & +a_0\alpha & +(a_0\alpha + a_1)\alpha & \\ \hline a_0 & a_0\alpha + a_1 & (a_0\alpha + a_1)\alpha + a_2 = a_0\alpha^2 + a_1\alpha + a_2. & \end{array}$$

Second. Assume the validity of the rule for $n = m$, and prove that its validity for $n = m + 1$ follows. Assume then that the rule carried out on

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$$

affords the remainder

$$a_0\alpha^m + a_1\alpha^{m-1} + \cdots + a_m = f(\alpha).$$

Now the polynomial of order $m + 1$ is

$$a_0x^{m+1} + a_1x^m + \cdots + a_mx + a_{m+1} = x \cdot f(x) + a_{m+1}.$$

Hence the next to the last remainder obtained by applying the rule to this polynomial would be $f(\alpha)$, since the succession of coefficients is the same for both polynomials up to a_{m+1}. By the rule the final remainder is obtained by multiplying the expression just obtained, in this case $f(\alpha)$, by α and adding the last coefficient, in this case a_{m+1}. This affords the final remainder

$$\alpha \cdot f(\alpha) + a_{m+1} = a_0\alpha^{m+1} + a_1\alpha^m + \cdots + a_m\alpha + a_{m+1}.$$

EXERCISES

1. Prove by complete induction that the partial remainders up to the final remainder obtained in the process of synthetic division are the coefficients of the quotient of $f(x)$ by $x - \alpha$.

2. Perform by synthetic division the following divisions.

(a) $x^3 - 7x^2 - 6x + 72 \div x - 4$.

Solution:

$$\begin{array}{rrrr|l} 1 - & 7 - & 6 + & 72 & \underline{4} \\ & 4 - & 12 - & 72 & \\ \hline 1 - & 3 - & 18 & 0 & \end{array}$$

Quotient $= x^2 - 3x - 18$.

(b) $x^3 - 9x + 10 \div x - 2$.

(c) $4x^3 - 7x - 87 \div x - 3$.

(d) $x^3 + 8x^2 - 4x - 32 \div x - 2$.

(e) $x^3 + 4x^2 - 7x - 30 \div x + 3$.

(f) $x^3 - 6x^2 + 11x - 6 \div x - 1$.

(g) $x^4 - 16x^3 + 86x^2 - 176x + 105 \div x^2 - 8x + 7$.

Hint. Since $x^2 - 8x + 7 = (x-7)(x-1)$, divide by $x - 7$ and the quotient by $x - 1$.

(h) $x^6 + 1 \div x + 1$.

(i) $x^9 - 1 \div x - 1$.

(j) $x^4 + x^3 - x - 1 \div x^2 - 1$.

(k) $x^5 - 2x^3 - 4x \div x - 3$.

(l) $x^6 - 2x^3 - 4x - 1 \div x + 2$.

(m) $4x^3 - 6x^2 - 2x - 1 \div x - 3$.

(n) $2x^4 + 5x^3 - 37x^2 + 44x + 84 \div x^2 + 5x - 6$.

163. Plotting of equations. We can now form the table of values necessary to plot an equation of the type

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = y.$$

Example. Plot $x^3 + 4x^2 - 4 = y$.

$$\begin{array}{rrrr|l} 1 + & 4 + & 0 - & 4 & \underline{1} \\ & +1 + & 5 + & 5 & \\ \hline 1 + & 5 + & 5 + & 1 & \end{array}$$

$$\begin{array}{rrrr|l} 1 + & 4 + & 0 - & 4 & \underline{-1} \\ & -1 - & 3 + & 3 & \\ \hline 1 + & 3 - & 3 - & 1 & \end{array}$$

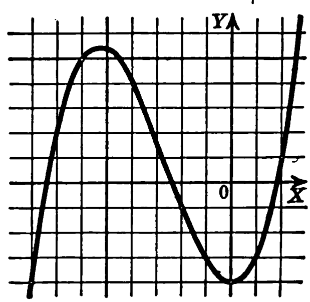
$$\begin{array}{rrrr|l} 1 + & 4 + & 0 - & 4 & \underline{-2} \\ & -2 - & 4 + & 8 & \\ \hline 1 + & 2 - & 4 + & 4 & \end{array}$$

$$\begin{array}{rrrr|l} 1 + & 4 + & 0 - & 4 & \underline{-3} \\ & -3 - & 3 + & 9 & \\ \hline & +1 - & 3 + & 5 & \end{array}$$

$$\begin{array}{rrrr|l} 1 + & 4 + & 0 - & 4 & \underline{-4} \\ & -4 + & 0 + & 0 & \\ \hline 1 + & 0 + & 0 - & 4 & \end{array}$$

x	y
0	-4
1	$+1$

x	y
-1	-1
-2	$+4$
-3	$+5$
-4	-4



In this figure two squares are taken to represent one unit of x. A single square represents a unit of y.

By an inspection of the figure it appears that the curve crosses the X axis at about $x = .8$, $x = -1.2$, and $x = -3.7$. Thus the equation for $y = 0$ has approximately these values for roots (§ 110).

164. Extent of the table of values. Since the object of plotting a curve is to obtain information regarding the roots of its equation, stretches of the curve beyond all crossings of the X axis are of no interest for the present purpose. Hence it is desirable to know when a table of values has been formed extensive enough to afford a plot which includes all the real roots. If for all values of x greater than a certain number the curve lies wholly above the axis, there are no real roots greater than that value of x.

By inspection of the preceding example it appears that if for a given value of x the signs of the partial remainders are all positive, thus affording a positive value of y, any greater value of x will afford only positive partial remainders and hence only positive values of y.

Thus *when all the partial remainders are positive no greater positive value of x need be substituted.*

Similarly, *when the partial remainders alternate in sign beginning with the coefficient of the highest power of x, no greater negative value of x need be substituted.*

In plotting, if the table of values consists of values that are large or are so distributed that the plot would not be well proportioned if one space on the paper were taken for each unit, a scale should be so chosen that the plot will be of good proportion, that is, so that all the portions of the curve between the extreme roots shall appear on the paper, and the curvatures shall not be too abrupt to form a graceful curve. This was done, for example, in the figure, § 163.

EXERCISES

Plot and measure the values of the real roots of the equations when $y = 0$.

1. $x^3 - 7x - 6 = y$.
2. $x^3 - 7x + 5 = y$.
3. $7x^3 - 9x - 5 = y$.
4. $x^3 - 31x + 19 = y$.
5. $x^3 - 12x - 14 = y$.
6. $4x^3 - 13x + 6 = y$.

7. $x^3 - 12x - 16 = y$.
8. $x^3 - 45x + 152 = y$.
9. $x^4 - 2x^3 - x + 2 = y$.
10. $8x^3 - 18x^2 + 17x - 6 = y$.
11. $x^4 - 17x^2 + x + 20 = y$.
12. $x^4 - 4x^3 + 9x^2 - 8x + 14 = y$.
13. $18x^3 - 36x^2 + 9x + 8 = y$.
14. $x^4 + 5x^3 + 12x^2 + 52x - 40 = y$.
15. $x^4 - 2x^3 - 7x^2 + 19x - 10 = y$.
16. $x^4 - 6x^3 + 3x^2 + 26x - 24 = y$.
17. $6x^4 - 13x^3 + 20x^2 - 37x + 24 = y$.

165. Roots of an equation. In the case of the linear and quadratic equations we have been able to find an explicit value of the roots in terms of the coefficients. Such processes are practically impossible in the case of most equations of higher degree. In fact the proof that any equation possesses a root lies beyond the scope of this book, and we make the

ASSUMPTION. *Every equation possesses at least one root.*

This is equivalent to the assumption that there is a number, rational, irrational, or complex, which satisfies any equation.

166. Number of roots. We determine the exact number of roots by the following

THEOREM. *Every equation of degree n has n roots.*

Given the equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$.

Let α_1 (see assumption) be a root of this equation. Then (p. 166) $x - \alpha_1$ is a factor of the left-hand member, and the quotient of $f(x)$ by $x - \alpha_1$ is a polynomial of degree $n - 1$. Suppose that

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - \alpha_1)(x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}).$$

By our assumption the quotient $x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1} = 0$ has at least one root, say α_2, to which corresponds the factor $x - \alpha_2$. Thus

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x^{n-2} + c_1x^{n-3} + \cdots + c_{n-2}).$$

Proceeding in this way we find successive roots and corresponding linear factors until the polynomial is expressed as the product of n linear factors as follows:

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) = 0,$$

where the roots are $\alpha_1, \alpha_2, \cdots, \alpha_n$.

REMARK. This theorem gives no information regarding how many of the roots may be real or imaginary. This depends on the particular values of the coefficients.

COROLLARY. *Any polynomial in x of degree n may be expressed as the product of n linear factors of the form $x - \alpha$, where α is a real or a complex number.*

It should be noted that the roots are not necessarily distinct. Several of the roots and hence several of the factors may be identical.

If $f(x)$ is divisible by $(x - \alpha_1)^2$, that is, if $\alpha_1 = \alpha_2$, we say that α_1 is a double root of the equation. Similarly, if $f(x)$ is divisible by $(x - \alpha_1)^r$, α_1 is called a multiple root of order r. When we say an equation has n roots we include each multiple root counted a number of times equal to its order.

THEOREM. *An equation of degree n has no more than n distinct roots.*

Let $f(x) = a_0x^n + \cdots + a_n = 0$ have the roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. Write the equation in the form

$$a_0(x - \alpha_1)\cdots(x - \alpha_n) = 0.$$

If r is a root distinct from $\alpha_1, \cdots, \alpha_n$, it must satisfy the equation and

$$a_0(r - \alpha_1)\cdots(r - \alpha_n) = 0.$$

Since this numerical expression vanishes one of its factors must vanish (§ 5). But $r \neq \alpha_1$, thus $r - \alpha_1 \neq 0$. Similarly, no one of the binomial factors vanishes. Thus (§ 5) $a_0 = 0$, which contradicts the hypothesis that the equation is of degree n.

This theorem may also be stated as follows:

COROLLARY I. *If an equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ of degree n is satisfied by more than n values of x, all its coefficients vanish.*

The proof of the theorem shows that if the equation has $n + 1$ roots, $a_0 = 0$. We should then have remaining an equation of degree $n - 1$, also satisfied by $n + 1$ values of x. Thus the coefficient of its highest power in x vanishes. Similarly, each of the coefficients vanishes.

COROLLARY II. *If two polynomials in one variable are equal to each other for every value of the variable, the coefficients of like powers of the variable are equal and conversely.*

Let $\quad a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n$

for every value of x.

Transpose, $\quad (a_0 - b_0)x^n + \cdots + a_n - b_n = 0.$

By Corollary I,
$$\begin{aligned} a_0 - b_0 &= 0, \text{ or } a_0 = b_0, \\ a_1 - b_1 &= 0, \text{ or } a_1 = b_1, \\ &\vdots \\ a_n - b_n &= 0, \text{ or } a_n = b_n. \end{aligned}$$

167. Graphical interpretation. The graphical interpretation of the theorems of the preceding section is that the graph of an equation of degree n cannot cross the X axis more than n times. Since each crossing of the X axis corresponds to a *real* root, there will be less than n crossings if the equation has imaginary roots.

168. Imaginary roots. We now show that imaginary roots occur in pairs. This we prove in the following

THEOREM. *If $a + ib$ is a root of an equation with real coefficients, $a - ib$ is also a root of the equation.*

If $a + ib$ is a root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, then $x - (a + ib)$ is a factor (p. 166). We wish to prove that $x - (a - ib)$ is also a factor, or what amounts to the same thing, that their product

$$\begin{aligned}[x - (a + ib)][x - (a - ib)] &= [(x - a) - ib][(x - a) + ib] \\ &= (x - a)^2 + b^2\end{aligned}$$

is a factor of $f(x)$. Divide $f(x)$ by $(x - a)^2 + b^2$ and we get

$$f(x) = Q(x)[(x - a)^2 + b^2] + rx + r', \tag{1}$$

where r and r' are real numbers. This remainder $rx + r'$ can be of no higher degree in x than the first, since the divisor

$$(x - a)^2 + b^2$$

is only of the second degree (§ 26). Now this equation (1) being an identity is true whatever value is substituted for x, as, for instance, the root of $f(x)$, $a + ib$. Substituting this value for x, we get

$$f(a+ib) = 0 = Q(a+ib)[(a+ib-a)^2+b^2]+r(a+ib)+r',$$

or (p. 33) and (p. 3) $\qquad 0 = 0 + ra + r' + irb,$

or (p. 156) $\qquad ra + r' = 0, \qquad (2)$

$$rb = 0. \qquad (3)$$

Since $b \neq 0$, by (3), p. 3, $r = 0$.

Also from (2), $\qquad r' = 0$.

Hence $\qquad rx + r' = 0$.

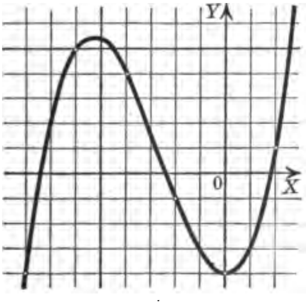
Consequently there is no remainder to the division of $f(x)$ by $(x-a)^2+b^2$, and hence if $a+ib$ is a root of $f(x)$, $a-ib$ is also a root.

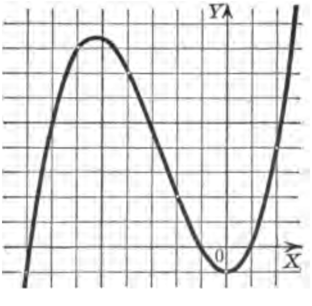
COROLLARY. *Every equation of odd degree with real coefficients has at least one real root.*

The roots cannot all be imaginary, else the degree of the equation would be even by the preceding theorem.

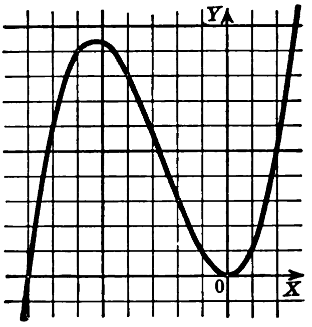
169. Graphical interpretation of imaginary roots. When we plot the equations

$$y = x^3 + 4x^2 - 4 \quad (1), \qquad y = x^3 + 4x^2 - 1 \quad (2),$$

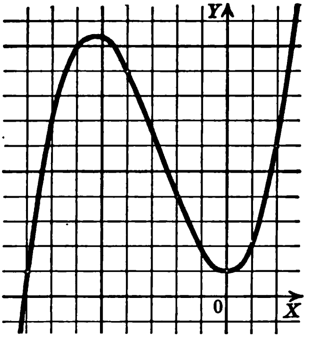




$y = x^3 + 4x^2$ (3),



$y = x^3 + 4x^2 + 1$ (4),



we see that corresponding to the increase of the constant term is a corresponding elevation of the curve with respect to the X axis. In fact in each case the curve is the same, but the value of y is gradually increased. In (1) and (2) we have three real roots, in (3) the curve touches the X axis, and in (4) we have only one real root. As the elbow of the curve is raised and fails to intersect the X axis *a pair* of roots cease to be real, and since a cubic equation always has three roots, *a pair* of roots become imaginary. Thus we have the

PRINCIPLE. *Corresponding to every elbow of the curve that does not intersect the X axis there is a pair of imaginary roots of the equation.*

The converse is not always true. It is not always possible to find as many elbows of the curve which do not meet the X axis as there are pairs of imaginary roots.

EXERCISES

Plot the following equations and determine from the plot how many roots are real.

1. $x^4 - 1 = y$.
2. $x^5 - 2 = y$.
3. $x^5 - x - 1 = y$.
4. $x^4 + 1 = y$.
5. $x^4 + x + 1 = y$.
6. $x^4 + 2x^2 + 2 = y$.
7. $x^3 - 3x^2 - x + 1 = y$.
8. $x^3 - 2x^2 + 4x - 1 = y$.
9. $2x^3 + 3x^2 + 5x + 6 = y$.
10. $x^3 - 3x^2 - 4x - 5 = y$.

170. Relation between roots and coefficients. If we write the expression (Corollary, p. 173)

$$x^n + b_1x^{n-1} + \cdots + b_n \equiv (x - \beta_1)(x - \beta_2)\cdots(x - \beta_n),$$

and multiply the factors, we obtain by equating coefficients of like powers of x (p. 174) relations between the roots and the coefficients. Take for example $n = 3$.

$$x^3 + b_1x^2 + b_2x + b_3 = (x - \beta_1)(x - \beta_2)(x - \beta_3)$$
$$= x^3 - (\beta_1 + \beta_2 + \beta_3)x^2 + (\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1)x - \beta_1\beta_2\beta_3 = 0.$$

Hence
$$b_1 = -(\beta_1 + \beta_2 + \beta_3),$$
$$b_2 = \beta_1\beta_2 + \beta_2\beta_3 + \beta_1\beta_3,$$
$$b_3 = -\beta_1\beta_2\beta_3.$$

This suggests the

THEOREM. *The coefficient of x^{n-1} is equal to the sum of the roots with their signs changed.*

The constant term is equal to the product of the roots with their signs changed.

In general the coefficient of x^{n-r} is equal to the sum of all possible products of r of the roots with their signs changed.

We prove this theorem by complete induction.

First. We have already established the theorem for equations of degree two on p. 106 and for equations of degree three above.

Second. Assume the theorem for $n = m$. That is, if

$$x^m + b_1x^{m-1} + \cdots + b_m \equiv (x - \beta_1)(x - \beta_2)\cdots(x - \beta_m), \quad (1)$$

we assume that b_r, the coefficient of x^{m-r}, is the sum of all possible products of r of the numbers $-\beta_1, -\beta_2, \cdots, -\beta_m$.

Multiply both sides of (1) by $x - \beta_{m+1}$. Denote the result by

$$x^{m+1} + b'_1x^m + \cdots + b'_{m+1} \equiv (x - \beta_1)(x - \beta_2)\cdots(x - \beta_{m+1}). \quad (2)$$

The term in x^{m+1-r} in this equation is obtained by multiplying the terms b_rx^{m-r} and $b_{r-1}x^{m+1-r}$ in (1) by x and $-\beta_{m+1}$ respectively. That is, in (2)

$$b'_r = b_r + b_{r-1}(-\beta_{m+1}). \quad (3)$$

Now all possible products of r of the quantities $-\beta_1, -\beta_2, \cdots, -\beta_{m+1}$ may be formed as follows: (1) Neglect $-\beta_{m+1}$, and form all possible products of r of those remaining. The sum of

these is b_r. (2) Form all possible products of $r-1$ of $-\beta_1$, $-\beta_2, \cdots, -\beta_m$, not including $-\beta_{m+1}$, and multiply each product by $-\beta_{m+1}$. Add all the products obtained. This process, it is observed, is precisely that indicated by (3).

Remark. It is noticed that in the rule the signs of the roots are always changed before forming any term. This does not involve any change when r is an even number, but is included in the rule for the sake of uniformity.

Corollary. *Every root of an equation is a factor of its constant term.*

171. The general term in the binomial expansion. On p. 129 we gave an expression for the $(r+1)$st term of the binomial expansion, the validity of which we now establish. In (1), § 170, let $\beta_1 = \beta_2 = \cdots = \beta_n$. Denote this common value by $-a$. The expression (1) becomes, on writing n in place of m,

$$x^n + b_1 x^{n-1} + \cdots + b_n = (x+a)^n.$$

By the theorem in § 170, b_r is the sum of all possible products of r of the negative roots. Since there are

$$C_{n,r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

such products, and since the roots are now identical, we obtain

$$\frac{n(n-1)\cdots(n-r+1)}{r!} x^{n-r} a^r$$

as the form of the $(r+1)$st term of the expansion of $(x+a)^n$.

172. Solution by trial. Since by the previous corollary every root of an equation is a factor of its constant term, we may in many cases test by synthetic division whether or not a given equation has integral roots. Thus the integral roots of the equation

$$x^4 - 8x^3 + 4x^2 + 24x - 21 = 0 \qquad (1)$$

must be factors of 21.

We try $+1$ by synthetic division,

$$\begin{array}{rrrrr|l} 1 & -8 & +4 & +24 & -21 & \underline{1} \\ & +1 & -7 & -3 & +21 & \\ \hline 1 & -7 & -3 & +21 & 0 & \end{array}$$

Thus 1 is a root of (1), and the quotient of the equation by $x-1$ is

$$x^3 - 7x^2 - 3x + 21 = 0. \qquad (2)$$

If this equation has any integral root it must be a factor of 21. We try $+3$ by synthetic division,

$$\begin{array}{rrrr|l} 1 & -7 & -3 & +21 & \underline{3} \\ & +3 & -12 & -45 & \\ \hline 1 & -4 & -15 & -24 & \end{array}$$

Thus 3 is not a root. We try $+7$,

$$\begin{array}{rrrr|l} 1 & -7 & -3 & +21 & \underline{7} \\ & +7 & +0 & -21 & \\ \hline 1 & +0 & -3 & 0 & \end{array}$$

Thus 7 is a root, and the remaining roots of (1) are the roots of

$$x^2 - 3 = 0,$$

that is, $$x = \pm\sqrt{3}.$$

Hence the roots of (1) are $+1$, $+7$, $\pm\sqrt{3}$.

EXERCISES

Solve by trial:

1. $x^3 - 7x^2 + 50 = 0$.

2. $x^3 - 9x + 28 = 0$.

3. $x^3 - 36x - 91 = 0$.

4. $x^3 + 9x + 26 = 0$.

5. $x^3 - 19x + 30 = 0$.

6. $x^3 - 27x - 54 = 0$.

7. $x^3 + 2x^2 - 23x + 6 = 0$.

8. $x^3 - 6x^2 + 11x - 6 = 0$.

9. $x^3 - 2x^2 - 11x + 12 = 0$.

10. $x^3 - 8x^2 + 19x - 20 = 0$.

11. $x^3 + 9x^2 + 27x + 26 = 0$.

12. $x^4 - 8x^3 + 8x^2 + 40x - 32 = 0$.

13. $x^4 - 13x^2 + 48x - 60 = 0$.

14. $x^4 - 3x^3 - 34x^2 + 18x + 168 = 0$.

15. $x^4 + 8x^3 - 7x^2 - 50x + 48 = 0$.

16. $x^4 - 3x^3 - 5x^2 + 29x - 30 = 0$.

17. $x^4 - 6x^3 + 13x^2 - 30x + 40 = 0$.

18. $x^4 - 8x^3 + 21x^2 - 34x + 20 = 0$.

19. $x^4 - 12x^3 + 43x^2 - 42x + 10 = 0$.

173. Properties of binomial surds. A **binomial surd** is a number of the form $a \pm \sqrt{b}$, where a and b are rational numbers, and where b is positive but not a perfect square.

Though we have not explicitly defined what we mean by the sum of an irrational number and a rational number, we shall assume that we can operate with the binomial surd just as we would be able to operate if b were a perfect square.

THEOREM I. *If a binomial surd* $a+\sqrt{b}=0$, *then* $a=0$ *and* $b=0$.

If $a+\sqrt{b}=0$ and either $a=0$ or $b=0$, clearly both must equal zero. Suppose, however, that neither a nor b equals zero. Then transposing we have $a=-\sqrt{b}$, and a rational number would be equal to an irrational number, which cannot be. Hence the only alternative is that both a and b equal zero.

THEOREM II. *If two binomial surds, as* $a+\sqrt{b}$ *and* $c+\sqrt{d}$, *are equal, then* $a=c$ *and* $b=d$.

Let $$a+\sqrt{b}=c+\sqrt{d}.$$

Transposing c, $$a-c+\sqrt{b}=\sqrt{d}. \tag{1}$$

Square and we obtain

$$(a-c)^2+b+2(a-c)\sqrt{b}=d,$$

or

$$(a-c)^2+b-d+2(a-c)\sqrt{b}=0.$$

Thus, by Theorem I, either $b=0$, which is contrary to the definition of a binomial surd, or $a-c=0$, that is, $a=c$. In the latter case (1) reduces to $\sqrt{b}=\sqrt{d}$, or $b=d$, and we have $a=c$ and $b=d$, which was to be proved.

$a+\sqrt{b}$ and $a-\sqrt{b}$ are called **conjugate binomial surds.**

THEOREM III. *If a given binomial surd* $a+\sqrt{b}$ *is the root of an equation with rational coefficients, then its conjugate is also a root of the same equation.*

The proof of this theorem, which should be performed in writing by each student, may be made analogously to the proof of the theorem on p. 174.

174. Formation of equations. If we know all the roots of an equation, we may form the equation in either one of two ways (see p. 167 and p. 177).

FIRST METHOD. *If* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are the given roots, multiply together the factors* $x-\alpha_1, \cdots, x-\alpha_n$.

SECOND METHOD. *From the given roots form the coefficients by the rule on* p. 177.

If the equation and all but one of its roots are known, that root can be found by the solution of a linear equation obtained from the coefficient of the second or the last term. If all but two of its roots are known, the unknown roots may be found by the solution of a pair of simultaneous equations formed from the same coefficients.

In the solution of the following exercises use is made of the theorem on p. 174, Theorem III, p. 180, and the various relations between the roots and the coefficients.

EXERCISES

1. Form the equations which have the following roots. Check the process by using both methods of § 174.

(a) $2, \ -3, \ 1.$

Solution:

First method. $(x-2)(x+3)(x-1) = x^3 - 7x + 6.$

Second method. Let the equation be

$$x^3 + b_1x^2 + b_2x + b_3 = 0.$$

Then, by § 170,

$$b_1 = -(2 - 3 + 1) = 0,$$
$$b_2 = -6 + 2 - 3 = -7,$$
$$b_3 = -2 \cdot 3 \cdot -1 = 6.$$

The equation then is $x^3 - 7x + 6 = 0.$

(b) $1, \ 2, \ 3.$

(c) $2, \ 2, \ 2, \ 2.$

(d) $3, \ 1, \ 1, \ 0.$

(e) $1, \ 0, \ 0, \ 0.$

(f) $\pm\sqrt{2}, \ \pm i.$

(g) $2, \ 4, \ -5.$

(h) $2, \ -3, \ 1, \ 0.$

(i) $2, \ 3, \ -5.$

(j) $7, \ \sqrt{5}, \ -\sqrt{5}.$

(k) $1, \ 2, \ -\frac{1}{2}, \ -\frac{3}{4}.$

(l) $3, \ 1+i, \ 1-i.$

(m) $-4, \ -3, \ 3 \pm \sqrt{5}.$

(n) $1 \pm i, \ -1 \pm i.$

(o) $2, \ \sqrt{-3}, \ -\sqrt{-3}.$

(p) $-1, \ 2, \ 3, \ -4.$

(q) $2\frac{1}{2}, \ 3\frac{1}{3}, \ -1\frac{1}{2}, \ -2\frac{1}{3}.$

(r) $\pm\sqrt{5}, \ \pm i\sqrt{7}.$

(s) $-5, \ 2+\sqrt{5}, \ 2-\sqrt{5}.$

(t) $\pm 1, \ \dfrac{\pm\sqrt{3}}{2}.$

(u) $\dfrac{1 \pm i\sqrt{3}}{2}, \ \dfrac{-1 \pm i\sqrt{3}}{2}.$

(v) $3, \ \dfrac{-3+\sqrt{5}}{2}, \ \dfrac{-3-\sqrt{5}}{2}.$

(w) $-1, \ \dfrac{1+i\sqrt{3}}{2}, \ \dfrac{1-i\sqrt{3}}{2}.$

2. The equation $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ has two roots -3 and $+1$. Find the remaining roots.

Solution: Let the unknown roots be a and b.

Then, by § 170, $\qquad -a - b + 3 - 1 = 2,$

$$-3ab = 12.$$

Solving for a and b, we obtain $\qquad a = -2$ or $+2$,

$$b = +2 \text{ or } -2.$$

3. $x^3 - 7x + 6 = 0$ has the roots 2 and 1. Find the remaining root.

4. $x^3 - 3x + 2 = 0$ has the root 1. Find the remaining roots.

5. $x^3 - 18x - 35 = 0$ has the root 5. Find the remaining roots.

6. Two roots of $x^4 - 35x^2 + 90x - 56 = 0$ are 1 and 2. Find the remaining roots.

7. The roots of $x^3 - 6x^2 - 4x + 24 = 0$ are in A.P. Find them.

8. The two equations $x^3 - 6x^2 + 11x - 6 = 0$ and $x^3 - 14x^2 + 63x - 90 = 0$ have a root common. Plot both equations on the same axes, and find all the roots of both equations.

9. Determine the middle term of the equation whose roots are -2, $+1$, 3, -4 without determining any other term.

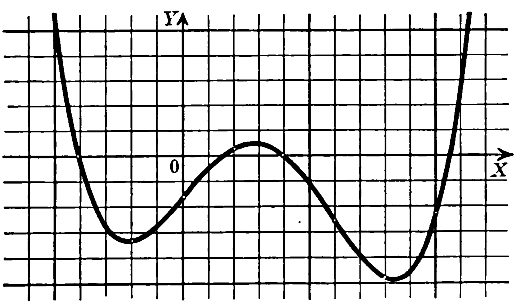
10. What is the last term of the equation whose roots are -4, 4, $\pm\sqrt{-3}$?

11. One root of $x^4 - 4x^3 + 5x^2 + 2x + 52 = 0$ is $3 - 2i$. Find the remaining roots.

12. One root of $x^4 - 4x^3 + 5x^2 + 8x - 14 = 0$ is $2 + i\sqrt{3}$. Find the others.

13. Plot the following equations, determine all the integral roots, and find the remaining roots by solving.

(a) $x^4 - 6x^3 + 24x - 16 = 0.$



x	y	x	y
0	-16	-1	-33
1	$+2$	-2	0
2	0	-3	$+155$
3	-25		
4	-48		
5	-21		
6	$+128$		

In this plot two squares on the X axis represent a unit of x, while one square on the Y axis represents ten units of y. The integral factors are $x - 2$ and $x + 2$, since ± 2 are roots, that is, are values of x for which the

curve is on the X axis. To find the quotient of our equation we first divide synthetically by 2, and then the quotient by -2, using the principle given in § 161.

$$\begin{array}{rrrrr|l} 1 - 6 + & 0 + & 24 - & 16 & \underline{2} \\ +2 - & 8 - & 16 + & 16 & \\ \hline 1 - 4 - & 8 + & 8 & 0 & \underline{-2} \\ -2 + & 12 - & 8 & & \\ \hline 1 - 6 + & 4 & 0 & & \end{array}$$

Thus the quotient of the polynomial and $(x-2)(x+2)$ is $x^2 - 6x + 4$. Solving the equation

$$x^2 - 6x + 4 = 0,$$

we obtain the two remaining roots, $x = 3 \pm \sqrt{5}$. These remaining roots might also be found by the method of exercise 2.

(b) $x^3 - 5x - 12 = 0.$ (c) $x^3 - 8x^2 + 7 = 0.$

(d) $x^3 - 7x^2 + 50 = 0.$ (e) $x^3 - 8x^2 + 13x - 6 = 0.$

(f) $x^3 - 6x^2 + 7x - 2 = 0.$ (g) $x^3 + 3x^2 + 4x - 24 = 0.$

(h) $x^4 - 3x^3 + 7x^2 - 21x = 0.$ (i) $x^4 - 3x^3 - 7x^2 + 27x - 18 = 0.$

(j) $x^4 - 9x^3 + 21x^2 - 19x + 6 = 0.$

(k) How many imaginary roots can an equation of the 5th degree have?

(l) $x^3 - ax^2 + bx - c = 0$ has two roots whose sum is zero. What is the third root? What are the two roots whose sum is zero?

(m) $x^3 + x^2 + bx + c = 0$ has one root the reciprocal of the other. What are the values of the roots?

(n) $x^3 - 4x^2 + ax + b^2 = 0$ has the sum of two roots equal to zero. What must be the values of a and b?

(o) $x^4 - 3x^3 + bx + 9 = 0$ has the sum of three of its roots equal to zero. What must be the value of b?

175. To multiply the roots by a constant. Suppose we have the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad (1)$$

whose roots are $\alpha_1, \alpha_2, \cdots, \alpha_n$. An equation of this type for values of n greater than 2 is usually not solvable by elementary methods. It often happens, however, that by changing its form slightly we may obtain an equation one or more of whose roots we can find. We shall see that if an equation has *rational* roots we may always find them if we change the form of the equation as indicated on the following page.

We seek to form from (1) an equation whose roots are equal to the roots of (1) multiplied by a constant factor, as k. Thus the equation we seek must have the roots $k\alpha_1$, $k\alpha_2$, $k\alpha_3$. We carry out the proof, which is perfectly general, on the equation of the third order

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

whose roots are α_1, α_2, α_3. The equation that we seek must have roots $k\alpha_1$, $k\alpha_2$, $k\alpha_3$. Since now (§ 53) $f(x) = 0$ is satisfied by α, where α stands for any one of the roots, that is, since $f(\alpha) \equiv 0$, evidently $f\left(\frac{z}{k}\right) = 0$ is satisfied by $k\alpha$, that is,

$$f\left(\frac{k\alpha}{k}\right) \equiv f(\alpha) \equiv 0.$$

Hence we obtain an equation that is satisfied by $k\alpha_1$, $k\alpha_2$, $k\alpha$, if in $f(x)$ we let $x = \frac{z}{k}$.

The required equation is then

$$f\left(\frac{z}{k}\right) = \frac{a_0z^3}{k^3} + \frac{a_1z^2}{k^2} + \frac{a_2z}{k} + a_3 = 0,$$

or, multiplying by k^3,

$$a_0z^3 + ka_1z^2 + k^2a_2z + k^3a_3 = 0.$$

This affords the general

RULE. *To multiply the roots of an equation by a constant k, multiply the successive coefficients beginning with the coefficient of x^{n-1} by k, k^2, $\cdots$, k^n respectively.*

In performing this operation the lacking powers of x should be supplied with zero coefficients.

EXAMPLE. Multiply the roots of $2x^3 - 3x + 4 = 0$ by 2.

Multiply the coefficients by the rule above,

$$2x^3 + 2 \cdot 0x^2 - 4 \cdot 3x + 8 \cdot 4 = 0.$$

Simplifying, $$x^3 - 6x + 16 = 0.$$

When an equation in form (2), p. 166, has fractional coefficients, an equation may be formed whose roots are a properly chosen multiple of the roots of the original equation and whose coefficients are integers.

COROLLARY I. *When k is a fraction this method serves to divide the roots of an equation by a given number.*

COROLLARY II. *When $k=-1$ this method serves to form an equation whose roots are equal to the roots of the original equation but opposite in sign. This is equivalent to the statement that $f(-x)=0$ has roots equal but opposite in sign to those of $f(x)=0$.*

EXERCISES

1. Form the equation whose roots are three times the roots of

$$x^4-6x^3-x+1=0.$$

Solution: Supplying the missing term in the equation, we have

$$x^4-6x^3+0x^2-x+1=0.$$

Since $k=3$, we have by the rule

$$x^4-3\cdot 6x^3+9\cdot 0x^2-27\cdot x^2+81=0,$$

or

$$x^4-18x^3-27x^2+81=0.$$

2. Find the equation whose roots are twice the roots of

$$x^4+3x^3-2x+4=0.$$

3. Find the equation whose roots are one half the roots of

$$x^3-2x^2+3x-4=0.$$

4. Find the equation whose roots are two thirds the roots of

$$x^3-4x-6=0.$$

5. By what may the roots of the following equations be multiplied so that in the resulting equation the coefficient of the highest power of x is unity and the remaining coefficients are integers? Form the equations.

(a) $3x^3-6x+2=0.$

Solution: We wish to bring into every term such a factor that all the resulting coefficients are divisible by 3.

Let $k=3.$

Supply the lacking term,

$$3x^3+0x^2-6x+2=0.$$

By rule, $3x^3+3\cdot 0x^2-9\cdot 6x+27\cdot 2=0.$

Dividing by 3, $x^3-18x+18=0.$

(b) $x^3 + \frac{x^2}{4} - \frac{x}{8} - 1 = 0.$ (c) $x^3 - \frac{1}{8} = 0.$

(d) $x^3 + \frac{a}{b}x^2 + \frac{a}{b^2}x + \frac{a}{b^3} = 0.$ (e) $x^4 + \frac{x^3}{3} - \frac{x^2}{9} + 1 = 0.$

(f) $2x^3 - 3x^2 - x + 4 = 0.$ (g) $3x^4 - 3x^2 - 4x + 1 = 0.$

(h) $x^4 - 6x^3 - 2x^2 + \frac{1}{4} = 0.$ (i) $16x^4 - 24x^3 + 8x^2 - 2x + 1 = 0.$

6. Form equations whose roots are the negatives of the roots of the following equations.

(a) $x^3 - 4x + 6 = 0.$

Solution: Supply the lacking term,

$$x^3 + 0x^2 - 4x + 6 = 0.$$

Changing signs we obtain by Corollary II

$$x^3 - 0x^2 - 4x - 6 = 0,$$

or

$$x^3 - 4x - 6 = 0.$$

(b) $x^3 - 2x^2 - 4x = 0.$ (c) $x^4 - 3x^2 + 1 = 0.$

(d) $x^4 - 2x^3 + x^2 + 2x - 1 = 0.$ (e) $x^3 + 3x^2 + 7x - 13 = 0.$

7. What effect does changing the sign of every term of the member involving x have on the graph of an equation?

8. What is the graphical interpretation of the transformation which changes the signs of the roots of an equation, that is, what relation does the graph of the equation before transformation bear to the graph of the equation after transformation (a) when the degree is an even number, (b) when the degree is an odd number?

9. If $4x^4 - 16x^3 - 85x^2 + 4x + 21 = 0$ has as two roots $-\frac{1}{2}$ and -3, what are the roots of $4x^4 + 16x^3 - 85x^2 - 4x + 21 = 0$?

10. If a root of $x^3 - 11x^2 + 36x - 36 = 0$ is 2, what are the roots of $x^3 + 11x^2 + 36x + 36 = 0$?

176. Descartes' rule of signs. A pair of successive like signs in an equation is called a **continuation** of sign. A pair of successive unlike signs is called a **change** of sign.

In the equation

$$2x^4 - 3x^3 + 2x^2 + 2x - 3 = 0 \tag{1}$$

are one continuation of sign and three changes of sign. This may be seen more clearly by writing merely the signs, + − + + −.

Let us now inquire what effect if any is noted on the number of changes of sign in an equation if the equation is multiplied by

a factor of the form $x - a$ when a is positive, that is, when the number of positive roots of the equation is increased by one. Let us multiply equation (1) by $x - 2$. We have then

$$\begin{array}{r}
2x^4 - 3x^3 + 2x^2 + 2x - 3 \\
x - 2 \\
\hline
2x^5 - 3x^4 + 2x^3 + 2x^2 - 3x \\
- 4x^4 + 6x^3 - 4x^2 - 4x + 6 \\
\hline
2x^5 - 7x^4 + 8x^3 - 2x^2 - 7x + 6
\end{array}$$

In this expression the succession of signs is $+ - + - - +$, in which there are *four* changes of sign, that is, one more change of sign than in (1). If an increase in the number of positive roots always brings about at least an equal increase in the number of changes of sign, there must be at least as many changes of sign in an equation as there are positive roots. This is the fact, as we now prove.

DESCARTES' RULE OF SIGNS. *An equation $f(x) = 0$ has no more real positive roots than $f(x)$ has changes of sign.*

ILLUSTRATION. In the equation of degree one $x - 2 = 0$ there is one change of sign and one real root. In the case of a linear equation there is no possibility of more than one change of sign. In the quadratic equation $x^2 + 2x + 1 = 0$ there is no change of sign, and also no positive root since for positive values of x the expression $x^2 + 2x + 1$ is always *positive* and hence never zero. In the equation $x^2 + 2x - 3 = 0$ we have one change of sign, and one positive root, $+1$.

We shall prove this general rule by complete induction.

First. We have just seen that the rule holds for an equation of degree one.

Second. We assume that the rule holds for an equation of degree m, and prove that its validity for an equation of degree $m + 1$ follows. We shall show that if we multiply an equation of degree m by $x - \alpha$, where α is positive, thus forming an equation of degree $m + 1$, the number of changes of sign in the new equation always exceeds the number of changes of sign in the

original equation by at least one. That is, the number of changes of sign increases at least as rapidly as the increase in the number of positive roots when such a multiplication is made.

Let $f(x) = 0$ represent any particular equation of the nth degree. The first sign of $f(x)$ is always $+$. The remaining signs occur in successive groups of $+$ or $-$ signs which may contain only one sign each. If any term is lacking, its sign is taken to be the same as an adjacent sign. Thus the most general way in which the signs of $f(x)$ may occur is represented in the following table, in which the dots represent an indefinite number of signs. The multiplication of $f(x)$ by $x - \alpha$ is represented schematically, only the signs being given.

	All + signs	All − signs	All + signs	All − signs	Further groups	All − signs
$f(x)$	$+ \cdots\cdot +$	$- \cdots\cdot -$	$+ \cdots\cdot +$	$- \cdots\cdot -$	$+ \cdots\cdot +$	$- \cdots\cdot -$
$x - \alpha$						$+ \quad -$
$xf(x)$	$+ + \cdots +$	$- - \cdots -$	$+ + \cdots +$	$- - \cdots -$	$+ + \cdots +$	$- - \cdots -$
$-\alpha f(x)$	$- \cdots -$	$- + \cdots +$	$+ - \cdots -$	$- + \cdots +$	$+ - \cdots -$	$- + \quad + +$
$(x-\alpha)f(x)$	$+ \pm \cdots \pm$	$- \pm \cdots \pm$	$+ \pm \cdots \pm$	$- \pm \cdots \pm$	$+ \pm \quad \pm$	$- \pm \cdots \pm +$

The $\pm$ sign indicates that either the $+$ or the $-$ sign may occur according to the value of the coefficients and of α. The vertical lines denote where changes of sign occur in $f(x)$. Assuming that all the ambiguous signs are taken so as to afford the fewest possible number of changes of sign, even then in $(x - \alpha)f(x)$ there is a change of sign at each or between each pair of the vertical lines, and in addition, one to the right of all the vertical lines. Thus as we increase the number of positive roots by one the number of changes of sign increases at least by one, perhaps by more.

The only possible variation that could occur in the succession of groups of signs in $f(x)$, namely, when the last group is a group of $+$ signs, does not alter the validity of the theorem.

We illustrate the foregoing proof by the following particular example.

Let $\quad f(x) = x^5 - 4x^3 - x + 2$, and let $a = 2$.

$$
\begin{array}{lrr}
\text{Multiply } f(x), & 1+0-4+0-1+2 & \text{4 changes} \\
\text{by} \quad x-2, & \underline{1-2} & \\
xf(x), & 1+0-4+0-1+2 & \\
-2f(x), & \underline{-2-0+8-0+2-4} & \\
(x-2)f(x), & 1-2-4+8-1+4-4 & \text{5 changes}
\end{array}
$$

177. Negative roots. Since $f(-x)$ has roots opposite in sign to those of $f(x)$ (p. 185), we can state

DESCARTES' RULE OF SIGNS FOR NEGATIVE ROOTS. *$f(x)$ has no more negative roots than there are changes in sign in $f(-x)$.*

If by Descartes' rule it appears that there cannot be more than a positive roots and b negative roots, and if $a + b < n$, the degree of the equation, then there must be imaginary roots, at least $n - (a + b)$ in number.

EXERCISES

1. Prove Descartes' rule of signs for $x^2 + bx + c = 0$ directly from the expression for b and c in terms of the roots (see § 115).

2. Find the maximum number of positive and negative roots and any possible information about imaginary roots in the following equations.

(a) $x^3 + 2x^2 + 1 = 0$.

Solution: Writing signs of $f(x)$, $++++$, there is no change, hence no positive root.

Writing signs of $f(-x)$, $-+++$, there is one change, hence no more than one negative root. Since there can be only one real root there must be two imaginary roots.

(b) $x^3 + 1 = 0$. (c) $x^4 - 2 = 0$.

(d) $x^3 - x + 1 = 0$. (e) $x^5 - x + 1 = 0$.

(f) $x^4 + x + 1 = 0$. (g) $x^5 + x^2 + 1 = 0$.

(h) $x^3 - 6x^2 + 4x - 1 = 0$. (i) $x^5 - 2x^4 - 3x^3 + 4x^2 + x + 1 = 0$.

(j) $x^5 + 2x^4 - 6x^3 - 4x^2 + x - 1 = 0$.

178. Integral roots. In finding the rational roots of an equation we make use of the following

THEOREM. *If the equation*

$$x^n + a_1x^{n-1} + \cdots + a_n = 0 \tag{1}$$

(where the a's are integers) has any rational root, such root must be an integer.

Suppose $\frac{p}{q}$ be a fraction reduced to its lowest terms which satisfies the equation.

Then $$\frac{p^n}{q^n} + \frac{a_1p^{n-1}}{q^{n-1}} + \cdots + a_n = 0$$
is an identity.

Then clearing of fractions and transposing,

$$p^n = -q(a_1p^{n-1} + \cdots + a_nq^{n-1}).$$

Thus some factor of q is a factor of p^n, that is, of p (p. 52), which contradicts the hypothesis that $\frac{p}{q}$ is reduced to its lowest terms.

Thus all the rational roots of the equation are integers, which as we know (§ 170) are factors of a_n.

179. Rational roots. If we seek the rational roots of

$$a_0x^n + \cdots + a_n = 0,$$

where $a_0 \neq 1$, we can multiply the roots by a properly chosen constant (§ 175) and obtain an equation of form (1) above whose integral roots may easily be found by synthetic division.

EXAMPLE. What rational roots, if any, has

$$3x^3 + 11x - 14 = 0? \tag{1}$$

Multiply the roots by 3, $3x^3 + 99x - 378 = 0.$

Divide by 3, $$x^3 + 33x - 126 = 0. \tag{2}$$

Since by Descartes' rule of signs equation (1) has only one positive root and no negative root, we do not need to carry the table further than to test for a positive root.

x	y
1	-92
2	-52
3	0

Form a table of values for equation (2) by synthetic division. We need only to try the factors of 126 (§ 170).

Thus (2) has the root 3. Hence the original equation (1) has the root $3 \div 3 = 1$.

RULE. *To find all the rational roots of an equation, transform the equation so that the first coefficient is* $+1$.

Find the maximum number of positive and negative roots by Descartes' rule of signs.

Find the integral roots of this equation by trial, and the roots of the original equation by dividing the integral roots found by the constant by which the roots were multiplied.

By the Theorem § 178 we are assured that *all* the rational roots can be found in this way.

EXERCISES

Find all the rational roots of the following equations.

1. $4x^3 = 27(x+1)$.
2. $15x^3 + 13x^2 - 2 = 0$.
3. $4x^3 - 5x - 6 = 0$.
4. $x^5 - 2\frac{5}{6}x^2 + 2\frac{3}{4}x - 1 = 0$.
5. $4x^3 - 8x^2 - x + 2 = 0$.
6. $3x^4 - 8x^3 - 36x^2 + 25 = 0$.
7. $4x^3 - 4x^2 + x - 6 = 0$.
8. $3x^3 + 13x^2 + 11x - 14 = 0$.
9. $4x^3 + 16x^2 - 9x - 36 = 0$.
10. $2x^3 - 21x^2 + 74x - 85 = 0$.
11. $6x^3 - 47x^2 + 71x + 70 = 0$.
12. $12x^3 - 52x^2 + 23x + 42 = 0$.
13. $6x^3 - 29x^2 + 53x - 45 = 0$.
14. $6x^4 - x^3 - 8x^2 - 14x + 12 = 0$.
15. $27x^3 + 63x^2 + 30x - 8 = 0$.
16. $2x^4 - 13x^3 + 16x^2 - 9x + 20 = 0$.
17. $3x^3 - 26x^2 + 52x - 24 = 0$.
18. $6x^4 - x^3 - 49x^2 + 55x - 50 = 0$.
19. $18x^3 + 81x^2 + 121x + 60 = 0$.
20. $12x^4 + 5x^3 - 23x^2 - 5x + 6 = 0$.
21. $10x^4 + 17x^3 - 16x^2 + 2x - 20 = 0$.
22. $9x^4 + 15x^3 - 143x^2 + 41x + 30 = 0$.
23. $36x^4 - 72x^3 - 31x^2 + 67x + 30 = 0$.
24. $26x^4 - 108x^3 + 323x^2 - 241x + 60 = 0$.

180. Diminishing the roots of an equation. In the preceding sections we have solved completely the problem of finding the rational roots of an equation. We now pass to the problem of

finding the approximate values of the irrational roots of an equation. In carrying out the process that we shall develop it is desirable to form an equation whose roots are equal respectively to the roots of the original equation each diminished by a constant.

Let $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \quad (1)$$

whose roots are $\alpha_1, \alpha_2, \cdots, \alpha_n$. Let a be any constant. We seek an equation whose roots are $\alpha_1 - a, \alpha_2 - a, \cdots, \alpha_n - a$.

If we let α stand for any one of the roots of (1), since $f(\alpha) = 0$ (p. 33), we see that

$$f(z + a) = 0 \text{ is satisfied by } \alpha - a,$$

that is, $$f(\alpha - a + a) = f(\alpha) = 0.$$

Thus to form the desired equation replace x by $z + a$. We obtain

$$f(x) = f(z + a) = a_0(z + a)^n + a_1(z + a)^{n-1} + \cdots + a_n = 0.$$

Developing each term by the binomial theorem and collecting like powers of z, we get an equation of the form

$$f(x) = F(z) = A_0z^n + A_1z^{n-1} + \cdots + A_n = 0, \quad (2)$$

where the A's involve the a's and a. This is the equation desired.

We now seek a convenient method of finding the values of the coefficients $A_0, A_1, A_2, \cdots, A_n$ when $a_0, a_1, a_2, \cdots, a_n$ are given numerically. Now A_n is the remainder from the division of $F(z)$ by z. But since $F(z) = f(x)$ and $z = x - a$, the remainder from dividing $F(z)$ by z is identical with the remainder from dividing $f(x)$ by $x - a$. Thus A_n is the remainder from dividing $f(x)$ by $x - a$. Furthermore, since A_{n-1} is the remainder from dividing $\dfrac{F(z) - A_n}{z}$ by z, it is also the remainder from dividing $\dfrac{f(x) - A_n}{x - a}$ by $x - a$. The process may be continued for finding the other A's. We may then diminish the roots of an equation by a as follows:

Rule. *The constant term of the new equation is the remainder from dividing $f(x)$ by $x - a$.*

The coefficient of z in the new equation is the remainder from dividing the quotient just obtained by $x - a$.

The coefficients of the higher powers of z are the remainders from dividing the successive quotients obtained by $x - a$.

EXAMPLE. Form the equation whose roots are 2 less than the roots of $x^4 - 2x^3 - 4x^2 + x - 1 = 0$.

The divisions required by the rule we carry out synthetically (p. 169).

$$
\begin{array}{llllll}
1 & -2 & -4 & +1 & -1 & \underline{|2} \\
  & +2 & +0 & -8 & -14 & \\
\hline
1 & +0 & -4 & -7 & \multicolumn{1}{|l}{-15} & \\
  & +2 & +4 & +0 & & \\
\hline
1 & +2 & +0 & \multicolumn{1}{|l}{-7} & & \\
  & +2 & +8 & & & \\
\hline
1 & +4 & \multicolumn{1}{|l}{+8} & & & \\
  & +2 & & & & \\
\hline
1 & +6 & & & &
\end{array}
$$

The desired equation is

$$x^4 + 6x^3 + 8x^2 - 7x - 15 = 0.$$

181. Graphical interpretation of decreasing roots. If an equation has roots a units less than those of another equation, if a is positive its intersections with the X axis or with any line parallel to the X axis are a units to the left of the corresponding intersections of the first equation. It is, in fact, the same curve, excepting that the Y axis is moved a units to the right. If a is negative, the Y axis is moved to the left.

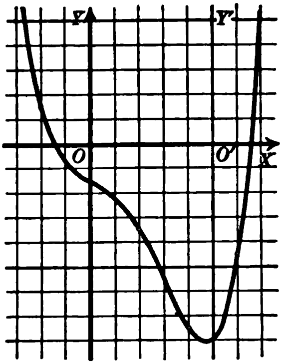
EXERCISES

Plot, decrease the roots by a units, and plot the new axes.

1. $x^4 - 3x^3 - 2x - 3 = 0$. $a = 3$. (1)

Solution:

$$
\begin{array}{llllll}
1 & -3 & -0 & -2 & -3 & \underline{|3} \\
  & +3 & -0 & -0 & -6 & \\
\hline
1 & -0 & -0 & -2 & \multicolumn{1}{|l}{-9} & \\
  & +3 & +9 & +27 & & \\
\hline
1 & +3 & +9 & \multicolumn{1}{|l}{+25} & & \\
  & +3 & +18 & & & \\
\hline
1 & +6 & \multicolumn{1}{|l}{+27} & & & \\
  & +3 & & & & \\
\hline
1 & +9 & & & &
\end{array}
$$



Thus the required equation is

$$x^4 + 9x^3 + 27x^2 + 25x - 9 = 0. \qquad (2)$$

x	y
0	-3
1	-7
2	-15
3	-9
-1	$+3$

In the figure one square on the Y axis represents two units of y, and two squares on the X axis represent one unit of x.

2. $x^4 - 16 = 0.$ $a = 2.$

3. $x^3 - 8 = 0.$ $a = 1.4.$

4. $x^4 - 2x^2 + 1 = 0.$ $a = .2.$

5. $x^3 + 4x - 8 = 0.$ $a = 3.$

6. $x^3 - 4x^2 - 2 = 0.$ $a = .5.$

7. $x^3 + 2x + 5 = 0.$ $a = -1.$

8. $x^3 + 4x^2 + x - 6 = 0.$ $a = -.4.$

9. $x^3 - 2x^2 + 8x - 7 = 0.$ $a = 2.$

10. $x^3 - 3x^2 + x - 1 = 0.$ $a = -.3.$

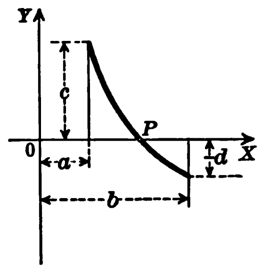
11. $x^4 - 3x^2 + 2x - 2 = 0.$ $a = -2.$

12. $x^3 - 15x^2 + 7x + 125 = 0.$ $a = 5.$

13. $2x^3 - 6x^2 + 4x - 3 = 0.$ $a = -3.$

14. $x^4 + 6x^3 + 10x^2 + x - 1 = 0.$ $a = -1.$

182. Location principle. If when plotting an equation $y = f(x)$ the value $x = a$ gives the corresponding value of y positive and equal to c, while the value $x = b$ gives the corresponding value of y negative, say equal to $-d$, then the point on the curve $x = a$, $y = c$ is above the X axis, and the point on the curve $x = b$, $y = -d$ is below the X axis. If our curve is unbroken, it must then cross the X axis at least once between the values $x = a$ and $x = b$, and hence the equation must have a root between those values of x. The shorter we can determine this interval a to b the more accurately we can find the root of the equation. This property of unbrokenness or **continuity** of the graph of $y = a_0x^n + a_1x^{n-1} + \cdots + a_n$ we assume. We assume then the following



Location principle. *When for two real unequal values of x, say $x = a$ and $x = b$, the value of $y = f(x)$ has opposite signs, the equation $f(x) = 0$ has a real root between a and b.*

Illustration. The equation $f(x) = x^3 + 3x - 5 = 0$ has a root between 1 and 2. Since $f(1) = -1$, $f(2) = 9$.

183. Approximate calculation of roots by Horner's method. We are now in a position to compute to any required degree of accuracy the real roots of an equation. Consider for example the equation

$$x^3 + 3x - 20 = 0. \tag{1}$$

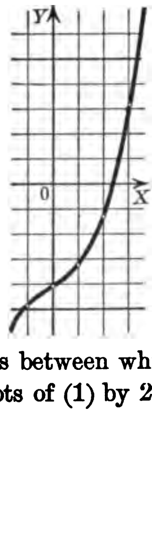
Form the table of values for plotting the equation

$$x^3 + 3x - 20 = y.$$

By the location theorem we find that a root is between $+2$ and $+3$. To find more precisely the position of the root we might estimate from the graph the position of the root and substitute say 2.3, 2.4, and so on, until we found two values between which the root lies. We can gain the same result with much less computation if we first diminish the roots of the equation so that the origin is at the less of the two integral values between which we know the root lies. Here we decrease the roots of (1) by 2,

x	y
0	-20
1	-16
2	-6
3	$+16$

x	y
-1	-24



$$\begin{array}{l}
1 + 0 + 3 - 20 \,\underline{|2}\\
\underline{ + 2 + 4 + 14}\\
1 + 2 + 7 \,|\, - \mathbf{6}\\
\underline{ + 2 + 8}\,|\\
1 + 4 \,|\, + \mathbf{15}\\
\underline{ + 2}\,|\\
\mathbf{1 + 6}
\end{array}$$

The equation whose roots are decreased by 2 is

$$x^3 + 6x^2 + 15x - 6 = 0. \tag{2}$$

We know that (2) has a root between 0 and 1, since equation (1) has a root between 2 and 3. From the graph we can estimate the position of the root. Having made an estimate, say .3, it is necessary to verify the estimate and determine by synthetic division precisely between which tenths the root lies. Thus, trying .3, we obtain

$$\begin{array}{r|l} 1 + 6.0 + 15.00 - 6.000 & .3 \\ +\ 0.3 + \ \ 1.89 + 5.067 & \\ \hline 1 + 6.3 + 16.89 - 0.933 & \end{array}$$

which shows that for $x = .3$ the curve is below the X axis, hence the root is greater than .3. But we are not justified in assuming that the root is between .3 and .4 until we have substituted .4 for x. This we proceed to do.

$$\begin{array}{r|l} 1 + 6.0 + 15.00 - 6.000 & .4 \\ +\ 0.4 + \ \ 2.66 + 7.064 & \\ \hline 1 + 6.4 + 17.66 + 1.064 & \end{array}$$

Since the value of y is positive for $x = .4$, the location principle shows that (2) has a root between .3 and .4, that is, (1) has a root between 2.3 and 2.4.

To find the root correct to two decimal places, move the origin up to the lesser of the two numbers between which the root is now known to lie. The new equation will have a root between 0 and .1. This process is performed as follows:

$$\begin{array}{rrrr|l} 1 + & 6.0 & + 15.00 & - 6.000 & .3 \\ + & 0.3 & + \ \ 1.89 & + 5.067 & \\ \hline 1 + & 6.3 & + 16.89 & -\mathbf{0.933} & \\ + & 0.3 & + \ \ 1.98 & & \\ \hline 1 + & 6.6 & +\mathbf{18.87} & & \\ + & 0.3 & & & \\ \hline 1 + & \mathbf{6.9} & & & \end{array}$$

Thus the new equation is

$$x^3 + 6.9\,x^2 + 18.87\,x - .933 = 0. \tag{3}$$

This equation has a root between 0 and .1. We can find an approximate value of the hundredths place of the root by solving

the linear equation $18.87\,x - .933 = 0$, obtained from (3) by dropping all but the term in x and the constant term.

Thus $$x = \frac{.933}{18.87} = .04.$$

This suggestion must be verified by synthetic division to determine between what hundredths a root of (3) actually lies.

$$\begin{array}{r} 1 + 6.90 + 18.870 - 0.9330 \,\underline{|.04} \\ + 0.04 + 0.277 + 0.7658 \\ \hline 6.94 + 19.147 - 0.1672 \end{array}$$

Thus the curve is below the X axis at $x = .04$ and hence the root is greater than .04. We must not assume that the root is between .04 and .05 without determining that the curve is above the X axis at $x = .05$.

$$\begin{array}{r} 1 + 6.90 + 18.870 - 0.9330 \,\underline{|.05} \\ + 0.05 + 0.347 + 0.9608 \\ \hline 6.95 + 19.217 + 0.0278 \end{array}$$

Thus the curve is above the X axis at $x = .05$. By the location principle (3) has a root between .04 and .05, that is, (1) has a root between 2.34 and 2.35. We say that the root 2.34 is correct to two decimal places. If a greater degree of precision is desired, the process may be continued and the root found correct to any required number of decimal places.

The foregoing process affords the following

RULE. *Plot the equation. Apply the location principle to determine between what consecutive positive integral values a root lies.*

Decrease the roots of the equation by the lesser of the two integral values between which the root lies.

Estimate from the plot the nearest tenth to which the root of the new equation lies, and determine by synthetic division precisely the successive tenths between which the root lies.

Decrease the roots of this equation by the lesser of the two tenths between which the root lies, and estimate the root to the nearest hundredth by solving the last two terms as a linear equation.

Determine by synthetic division precisely the hundredths interval in which the root must lie.

Proceed similarly to find the root correct to as many places as may be desired.

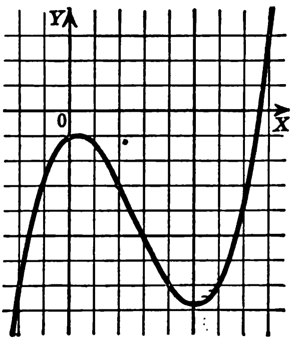
The sum of the integral, tenths, and hundredths values next less than the root in the various processes is the approximate value of the root.

To find the negative roots of an equation $f(x)=0$, find the positive roots of $f(-x)=0$ and change their signs.

When all the roots are real a check to the accuracy of the computation may be found by adding the roots together. The result should be the coefficient of the second term.

EXERCISES

Find the values of the real roots of the following equations correct to three decimal places.



x	y
0	-1
1	-3
2	-7
3	-7
4	$+3$

1. $x^3+4x^2+x+1=0.$ (1)

Solution: Since by Descartes' rule of signs there are no positive roots, we form the equation $f(-x)=0$ and seek its positive root.

Thus

$$x^3-4x^2+x-1=0. \qquad (2)$$

Plot the equation (2) set equal to y. In the figure two squares are taken as the unit of x.

There is a root of this equation between 3 and 4.

Decrease the roots of (2) by 3,

$$\begin{array}{llll|l}
1 & -4 & +1 & -1 & \underline{3} \\
 & +3 & -3 & -6 & \\
\hline
1 & -1 & -2 & -7 & \\
 & +3 & +6 & & \\
\hline
1 & +2 & +4 & & \\
 & +3 & & & \\
\hline
1 & +5 & & &
\end{array}$$

The equation is $$x^3 + 5x^2 + 4x - 7 = 0. \tag{3}$$
From the plot we estimate the root of (3) at .8.

Verify,
$$\begin{array}{rrrr|l} 1 + & 5.0 + & 4.00 - & 7.000 & \underline{.8} \\ & + 0.8 + & 4.64 + & 6.912 & \\ \hline & + 5.8 + & 8.64 - & 0.088 & \end{array}$$
$$\begin{array}{rrrr|l} 1 + & 5.0 + & 4.00 - & 7.000 & \underline{.9} \\ & + 0.9 + & 5.31 + & 8.379 & \\ \hline & + 5.9 + & 9.31 + & 1.379 & \end{array}$$

Thus the root is determined between .8 and .9.
Decrease the roots of (3) by .8,
$$\begin{array}{rrrr|l} 1 + & 5.0 + & 4.00 & - 7.000 & \underline{.8} \\ & + 0.8 + & 4.64 & + 6.912 & \\ \hline 1 + & 5.8 + & 8.64 & - .088 & \\ & + 0.8 + & 5.28 & & \\ \hline 1 + & 6.6 & + 13.92 & & \\ & + 0.8 & & & \\ \hline 1 + & 7.4 & & & \end{array}$$

The equation is $$x^3 + 7.4x^2 + 13.92x - .088 = 0. \tag{4}$$

Estimate the root of (4) at $x = \dfrac{.088}{13.92} = .006.$

Verify,
$$\begin{array}{rrrr|l} 1 + & 7.400 + & 13.920000 - & .088000 & \underline{.006} \\ & + 0.006 + & 00.044436 + & .083784 & \\ \hline & + 7.406 + & 13.964 & - .004216 & \end{array}$$
$$\begin{array}{rrrr|l} 1 + & 7.400 + & 13.920000 - & .088000 & \underline{.007} \\ & + 0.007 + & 00.051849 + & .097804 & \\ \hline & + 7.407 + & 13.972 & + .009804 & \end{array}$$

Thus the root of the equation (1) correct to three decimal places is -3.806.

2. $x^3 - 4 = 0.$

3. $x^4 - 3 = 0.$

4. $x^3 + x = 20.$

5. $3x^4 - 5x^3 = 31.$

6. $x^3 + x^2 = 100.$

7. $x^3 - x - 33 = 0.$

8. $x^4 + x - 100 = 0.$

9. $x^3 - 8x - 24 = 0.$

10. $x^4 - 4x^3 + 12 = 0.$

11. $x^4 + x^2 + x = 111.$

12. $x^3 - x^2 + x - 44 = 0.$

13. $x^3 + 10x - 13 = 0.$

14. $x^3 + 3x^2 - 2x - 1 = 0.$

15. $x^3 + x^2 + x - 99 = 0.$

16. $x^3 - 9x^2 - 2x + 101 = 0.$

17. $x^4 + x^3 + x^2 - 88 = 0.$

18. $x^4 - 12x^3 - 16x + 41 = 0.$

19. $x^3 - 6x^2 + 5x + 11 = 0.$

20. $x^3 - 10x^2 + 35x + 50 = 0.$

21. $2x^4 - 4x^3 + 3x^2 - 1 = 0.$

22. $3x^4 - 2x^3 - 21x^2 - 4x + 11 = 0.$

23. $9x^3 - 45x^2 + 34x + 37 = 0.$

184. Roots nearly equal. Suppose we wish to find the positive roots, if any exist, of

$$x^3 + 17x^2 - 46x + 29 = 0. \tag{1}$$

x	y
0	+ 29
1	+ 1
2	+ 13
3	+ 71

By Descartes' rule of signs we see that there can be only two positive roots. We obtain the adjacent table of values. From the plot that these values indicate we cannot tell whether any real root exists between 1 and 2, but if it does exist the plot indicates that it is nearer 1 than 2.

Decrease the roots of (1) by 1,

$$\begin{array}{rrrr|l}
1 & +17 & -46 & +29 & \underline{1} \\
 & +1 & +18 & -28 & \\ \hline
1 & +18 & -28 & \multicolumn{1}{|r}{+1} & \\
 & +1 & +19 & & \\ \hline
1 & +19 & \multicolumn{1}{|r}{-9} & & \\
 & +1 & & & \\ \hline
1 & +20 & & &
\end{array}$$

The new equation is

$$x^3 + 20x^2 - 9x + 1 = 0. \tag{2}$$

Estimate the root of (2) at .2 and carry the origin up to .2 and also up to .3.

$$\begin{array}{rrrr|l}
1 & +20.0 & -9.00 & +1.000 & \underline{.2} \\
 & +0.2 & +4.04 & -0.992 & \\ \hline
1 & +20.2 & -4.96 & \multicolumn{1}{|r}{+\mathbf{0.008}} & \\
 & +0.2 & +4.08 & & \\ \hline
1 & +20.4 & \multicolumn{1}{|r}{-\mathbf{0.88}} & & \\
 & +0.2 & & & \\ \hline
1 & +\mathbf{20.6} & & &
\end{array}
\qquad
\begin{array}{rrrr|l}
1 & +20.0 & -9.00 & +1.000 & \underline{.3} \\
 & +0.3 & +6.09 & -0.873 & \\ \hline
1 & +20.3 & -2.91 & \multicolumn{1}{|r}{+\mathbf{0.127}} & \\
 & +0.3 & +6.18 & & \\ \hline
1 & +20.6 & \multicolumn{1}{|r}{+\mathbf{3.27}} & & \\
 & +0.3 & & & \\ \hline
1 & +\mathbf{20.9} & & &
\end{array}$$

By Descartes' rule of signs on the numbers obtained by moving the origin to .3, it is seen that there are no positive roots of (2) greater than .3, while the rule would indicate that there might be roots greater than .2. We consider the equation

$$x^3 + 20.6x^2 - 0.88x + 0.008 = 0. \tag{3}$$

Estimate the roots of (3) at

$$x = \frac{0.008}{0.88} = .009.^*$$

Verify,

$$\begin{array}{l} 1 + 20.60 - 0.880 + 0.00800 \underline{|.01} \\ \underline{ + 0.01 + 0.206 - 0.00674} \\ 1 + 20.61 - 0.674 + 0.00126 \end{array}$$

$$\begin{array}{l} 1 + 20.60 - 0.880 + 0.00800 \underline{|.02} \\ \underline{ + 0.02 + 0.412 - 0.00936} \\ 1 + 20.62 - 0.468 - 0.00136 \end{array}$$

This determines a root of (3) between .01 and .02.

$$\begin{array}{l} 1 + 20.60 - 0.880 + 0.00800 \underline{|.03} \\ \underline{ + 0.03 + 0.619 - 0.00483} \\ 1 + 20.63 - 0.161 + 0.00317 \end{array}$$

This determines another root of (3) between .02 and .03.

Decrease the roots of (3) by .01,

$$\begin{array}{llll} 1 + 20.60 & - 0.880 & + 0.00800 & \underline{|.01} \\ + 0.01 & + 0.206 & - 0.00674 & \\ \hline 1 + 20.61 & - 0.674 & + \mathbf{0.00126} & \\ + 0.01 & + 0.206 & & \\ \hline 1 + 20.62 & - \mathbf{0.468} & & \\ + 0.01 & & & \\ \hline 1 + \mathbf{20.63} & & & \end{array}$$

The new equation is

$$x^3 + 20.63\,x^2 - 0.468\,x + 0.00126 = 0. \tag{4}$$

Estimate the root of (4) at $x = \dfrac{0.00126}{0.468} = .002.^*$

Verify,

$$\begin{array}{l} 1 + 20.630 - 0.468 + 0.00126 \underline{|.002} \\ \underline{ + 0.002 + 0.041 - 0.00085} \\ 1 + 20.632 - 0.427 + 0.00041 \end{array}$$

$$\begin{array}{l} 1 + 20.630 - 0.468 + 0.00126 \underline{|.003} \\ \underline{ + 0.003 + 0.062 - 0.00122} \\ 1 + 20.633 - 0.406 + 0.00004 \end{array}$$

* **We observe that in these two cases the estimated values of the roots are shown by the verification to be inaccurate. This should insure great care in making the verification. *The estimated values should never be assumed to be accurate without verification***

This indicates that a root is between .003 and .004.

Verify,
$$\begin{array}{r|l} 1 + 20.630 - 0.468 + 0.00126 & .004 \\ + \quad 0.004 + 0.083 - 0.00154 & \\ \hline 1 + 20.634 - 0.385 - 0.00028 & \end{array}$$

This determines a root of (3) between .003 and .004.
Thus one root of (1) correct to three decimal places is 1.213.
The other root could be found similarly to be 1.229.

EXERCISES

Find all the real roots of the following equations correct to four decimal places.

1. $x^3 - 7x + 7 = 0.$
2. $7x^3 - 8x^2 - 14x + 16 = 0.$
3. $2x^5 - 4x^3 - 3x^2 + 6 = 0.$
4. $4x^4 - 5x^3 - 8x + 10 = 0.$
5. $3x^3 - 10x^2 - 33x + 110 = 0.$

CHAPTER XVIII

DETERMINANTS

185. Solution of two linear equations. We have already treated the solution of linear equations in two variables and stated (p. 47) the method of solving three or more linear equations in three or more variables. This latter process is rather laborious and can be very much abridged and also developed more symmetrically by the considerations of the present chapter. Let us solve the equations

$$a_1x + b_1y = c_1, \tag{1}$$

$$a_2x + b_2y = c_2. \tag{2}$$

Multiply (1) by b_2 and (2) by b_1, and we obtain

$$\begin{array}{r} a_1b_2x + b_1b_2y = b_2c_1 \\ \underline{a_2b_1x + b_1b_2y = b_1c_2} \end{array}$$

Subtracting, we get $(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2,$

or if $a_1b_2 - a_2b_1 \neq 0,$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}. \tag{3}$$

Similarly, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{4}$$

We note that the denominators of the expressions for x and y are the same. This denominator we will denote symbolically by the following notation:

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{5}$$

The symbol in the right-hand member is called a **determinant.** Since there are two rows and two columns, this determinant is said to be of the second order. The left-hand member of the equation is called the **development** of the determinant. The symbols a_1, b_1, a_2, b_2 are called **elements** of the determinant, while the elements a_1 and b_2 are said to comprise its **principal diagonal.**

RULE. *The development of any determinant of the second order is obtained by subtracting from the product of the elements on the principal diagonal the product of the elements on the other diagonal.*

Thus $\begin{vmatrix} x & y \\ x_1 & y_1 \end{vmatrix} = xy_1 - x_1y$; $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2.$

Evidently each term of the development contains one and only one element of each row and each column, that is, for instance, the letter a and the subscript 1 appear in each term of (5) once and only once.

We can now rewrite the solution (3) and (4) of equations (1) and (2) in determinant form:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{6}$$

It is noted that the numerator of the expression for x is formed from the denominator by replacing the column which contains the coefficients of x, a_1 and a_2, by the constant terms c_1 and c_2. Similarly, in the numerator of the expression for y, b_1 and b_2 of the denominator are replaced by c_1 and c_2.

One should keep in mind that a determinant is merely a symbolic form of expression for its development. In the case of determinants of the second order the introduction of the new notation is hardly necessary, as the development itself is simple; just as we should scarcely need to introduce the exponential notation if we had to consider only the squares of numbers. It turns out, however, as we shall see, that we are able to denote by determinants with more than two rows and columns expressions with whose development it would be very laborious to deal.

186. Solution of three linear equations. Let us solve the equations

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$

$$a_2x + b_2y + c_2z = d_2, \tag{2}$$

$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

Eliminating y from (1) and (2) and (1) and (3), we obtain

$$(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2)z = d_1b_2 - d_2b_1,$$
$$(a_3b_1 - a_1b_3)x + (c_3b_1 - b_3c_1)z = d_3b_1 - d_1b_3.$$

Eliminating z,

$$\begin{aligned}[(a_1b_2 - a_2b_1)(c_3b_1 - b_3c_1) - (a_3b_1 - a_1b_3)(b_2c_1 - b_1c_2)]x \\ = (d_1b_2 - d_2b_1)(c_3b_1 - b_3c_1) - (d_3b_1 - d_1b_3)(b_2c_1 - b_1c_2).\end{aligned}$$

Developing, canceling, and dividing by b_1, we obtain

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_3b_2c_1 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3}. \qquad (4)$$

Following the analogy of the last section, we write the denominator

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (5)$$

The right-hand member of this equation we call a determinant of the **third** order, and the left-hand member its development. As in the determinant of the second order, the elements a_1, b_2, c_3 comprise the principal diagonal; each term of the development contains one and only one element of each row and each column, and all possible terms so constructed are included in the development. The signs of the terms of the determinant of the third order may be kept in mind by the following device.

Rewrite the first and second columns to the right of the determinant as follows:

$$\left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right|.$$

The positive terms are found on the diagonals running down from left to right, the negative terms on the diagonals running up from left to right.

The numerator of the fraction (4) expressed in determinant notation is

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

We can find similarly the values of y and z that satisfy equations (1), (2), and (3). The solutions of the equations in determinant form are as follows:

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}};\quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}};\quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{6}$$

The same principle observed on p. 204 for forming the determinants in the numerators of the expressions for x and y may be followed here. The determinant in the numerator of the expression for x, y, or z is found from the denominator by replacing the column that contains the coefficients of the variable in question by a column consisting of constant terms. Thus in the numerator of z we find the column d_1, d_2, d_3 replacing the column c_1, c_2, c_3 of the denominator.

EXERCISES

1. Find the value of the following determinants.

(a) $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 6 & 2 \\ 1 & 0 & 1 \end{vmatrix}$.

Solution: $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 6 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 18 + 4 + 0 - 6 - 8 - 0 = 8.$

(b) $\begin{vmatrix} 4 & 3 & 1 \\ 1 & 2 & 0 \\ 6 & 1 & 1 \end{vmatrix}$.

(c) $\begin{vmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$.

(d) $\begin{vmatrix} 3 & 4 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 6 \end{vmatrix}$.

(e) $\begin{vmatrix} 2 & 1 & 1 \\ 6 & 3 & 3 \\ 4 & 2 & 3 \end{vmatrix}$.

(f) $\begin{vmatrix} a & x & y \\ 0 & b & c \\ 0 & c & b \end{vmatrix}$.

(g) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

(h) $\begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix}$.

(i) $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$.

2. Solve the following equations by determinants.

(a) $\begin{aligned} 2x + 3y &= 4, \\ x - 2y &= 1. \end{aligned}$

Solution: Using the expressions (6), p. 206, we obtain

$$x = \frac{\begin{vmatrix} 4 & 3 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}} = \frac{-8-3}{-4-3} = \frac{11}{7}, \quad y = \frac{\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}} = \frac{2-4}{-4-3} = \frac{2}{7}.$$

Check: $\frac{11}{7} - \frac{4}{7} = \frac{7}{7} = 1.$

(b) $\begin{aligned} 2x + 5y &= 1, \\ 7x + 6y &= 2. \end{aligned}$ (c) $\begin{aligned} 2x + 7y &= 1, \\ 3x - 9y &= 2. \end{aligned}$ (d) $\begin{aligned} x + 4y &= 2, \\ 2x - 3y &= -1. \end{aligned}$ (e) $\begin{aligned} x - 7y &= 12, \\ 7x + y &= 11. \end{aligned}$

3. Solve the following equations by determinants.

(a) $\begin{aligned} x + y + z &= 2, \\ x + 3y - 4 &= 0, \\ y - 2z &= 6. \end{aligned}$

Solution: Rearranging so that terms in the same variable are in a column, and supplying the zero coefficients, we get

$$\begin{aligned} x + y + z &= 2, \\ x + 3y + 0z &= 4, \\ 0x + y - 2z &= 6. \end{aligned}$$

By (6), p. 206,
$$x = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 0 \\ 6 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & -2 \end{vmatrix}} = \frac{-12+0+4-18+8+0}{-6+0+1-0-0+2} = \frac{-18}{-3} = 6,$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 1 & 4 & 0 \\ 0 & 6 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & -2 \end{vmatrix}} = \frac{-8+0+6-0+4+0}{-3} = \frac{2}{-3} = -\frac{2}{3},$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 1 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & -2 \end{vmatrix}} = \frac{18+0+2-0-6-4}{-3} = \frac{10}{-3} = -\frac{10}{3}.$$

Check: $6 + (-\frac{2}{3}) + (-\frac{10}{3}) = 6 - \frac{12}{3} = 6 - 4 = 2.$

(b) $\begin{aligned} 2x + 3y &= 12, \\ 3x + 2z &= 11, \\ 3y + 4z &= 10. \end{aligned}$

(c) $\begin{aligned} \tfrac{1}{3}x - \tfrac{1}{2}y &= 0, \\ \tfrac{1}{3}x - \tfrac{1}{2}z &= 1, \\ \tfrac{1}{2}z - \tfrac{1}{3}y &= 2. \end{aligned}$

(d) $\begin{aligned} x + y - z &= 17, \\ x + z - y &= 13, \\ y + z - x &= 7. \end{aligned}$

(e) $\begin{aligned} 2x + 2y &= 7, \\ 7x + 9z &= 29, \\ y + 8z &= 17. \end{aligned}$

(f) $\begin{aligned} x + y + z &= 100, \\ 3x - 2z &= 4, \\ 5y - 4z &= 0. \end{aligned}$

(g) $\begin{aligned} x + y + 2z &= 34, \\ x + 2y + z &= 33, \\ 2x + y + z &= 32. \end{aligned}$

(h) $\begin{aligned} x + y + z &= 9, \\ x + 2y + 3z &= 14, \\ x + 3y + 6z &= 20. \end{aligned}$

(i) $\begin{aligned} ax + by - cz &= 2ab, \\ by + cz - ax &= 2bc, \\ cz + ax - by &= 2ac. \end{aligned}$

(j) $\begin{aligned} .2x + .3y + .4z &= 29, \\ .3x + .4y + .5z &= 38, \\ .4x + .5y + .6z &= 51. \end{aligned}$

(k) $\begin{aligned} 3x + 2y + 3z &= 110, \\ 5x + y - 4z &= 0, \\ 2x - 3y + z &= 0. \end{aligned}$

187. Inversion. In order to find the development of determinants with more than three rows and columns, the idea of an inversion is necessary. If in a series of positive integers a greater integer precedes a less, there is said to be an **inversion.** Thus in the series 1 2 3 4 there is no inversion, but in the series 1 2 4 3 there is one inversion, since 4 precedes 3. In 1 4 2 3 there are two inversions, as 4 precedes both 2 and 3; while in 1 4 3 2 there are three inversions, since 4 precedes 2 and 3, and also 3 precedes 2.

188. Development of the determinant. In the development of the determinant of order three we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2. \quad (1)$$

If we keep the order of letters in each term the same as their order in the principal diagonal (as we have done in the development above), it is observed that the subscripts in the various terms take on all possible permutations of the three digits 1, 2, and 3. The permutations that occur in the positive terms are 1 2 3, 2 3 1, 3 1 2, in which occur respectively 0, 2, and 2 inversions. The permutations that occur in the subscripts of the negative terms are 3 2 1, 2 1 3, 1 3 2, in which occur respectively 3, 1, and 1 inversions.

Thus in the subscripts of the positive terms an even number of inversions occur, while in the subscripts of the negative terms an odd number of inversions occur. This means of determining the sign of a term of the development we shall assume in general.

When we have a determinant with n rows and columns it is called a **determinant of the nth order**. The development of such a determinant is defined by the following

RULE. *The development of a determinant of the nth order is equal to the algebraic sum of the terms consisting of letters following each other in the same order in which they are found in the principal diagonal but in which the subscripts take on all possible permutations. A term has the positive or the negative sign according as there is an even or an odd number of inversions in the subscripts.*

This means of finding the development of a determinant is useful in practice only when the elements of the determinant are letters with subscripts such as in (2) below. When the elements are numbers we shall find the value of the determinant by a more convenient method.

In this statement it is assumed that the number of inversions in the subscripts of the principal diagonal is zero. If this number of inversions is not zero, the sign of any term is + or − according as the number of inversions in its subscripts differs from the number in the subscripts of the principal diagonal by an even or an odd number.

Since each term contains every letter $a, b, \cdots, k$ and also every index $1, 2, \cdots, n$, one element of each row and column occurs in each term.

In the determinant of the fourth order

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \tag{2}$$

the terms $a_2b_4c_1d_3$ and $a_4b_2c_3d_1$, for instance, have the minus sign, as 2 4 1 3 has three inversions and 4 2 3 1 has five inversions; while the terms $a_1b_4c_2d_3$ and $a_4b_3c_2d_1$ have two and six inversions respectively and hence have the positive sign.

189. Number of terms. We apply the theorem of permutations to prove the following

THEOREM. *A determinant of the nth order has n! terms in its development.*

Since the number of terms is the same as the number of permutations of the n indices taken all at a time, the theorem follows immediately from the corollary on p. 145.

190. Development by minors. In the development of the determinant of order three, p. 208, we may combine the terms as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}: \qquad (1)$$

We observe that the coefficient of a_1 is the determinant that we obtain by erasing the row and column in which a_1 lies. A similar fact holds for the coefficients of a_2 and a_3. The determinant obtained by erasing the row and column in which a given element lies is called the **minor** of that element. Thus $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ is the minor of a_2. We notice that in the above development by minors (1) the sign of a given term is + or − according as the sum of the number of the row and the number of the column of the element in that term is even or odd. Thus in the first term a_1 is in the first row and the first column, and since $1 + 1 = 2$, the statement just made is verified for that case. Similarly, a_2 is in the first column and the second row, and since $1 + 2 = 3$ is odd, the sign is minus and the law holds here. The last term is positive, which we should expect since a_3 is in the first column and the third row, and $1 + 3 = 4$. The proof for the general validity of this law of signs is found on p. 215.

The elements of any other row or column than the first may be taken and the development given in terms of the minors with

respect to such elements. For instance, take the development with respect to the elements of the second row,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

The rule of signs is the same as given above; that is, for instance, the last term is negative, as c_2 is in the third column and the second row, and $2 + 3 = 5$. By generalizing these considerations we may find the development of a determinant by minors by the following

RULE. *Write in succession the elements of any row or column, each multiplied by its minor.*

Give each term a + or a − sign according as the sum of the number of the row and the number of the column of the element in that term is even or odd.

Develop the determinant in each term by a similar process until the value of the development can be determined directly by multiplication.

That this rule for development gives the same result as the definition given in § 188 we have seen for a determinant of order three. The fact holds in general, as we shall prove (p. 215).

EXERCISES

1. In the determinant of order four on p. 209 what sign should be prefixed to the following terms?

(a) $c_4b_3a_2d_1$.

Solution: $c_4b_3a_2d_1 = a_2b_3c_4d_1$. In 2 3 4 1 there are three inversions. The sign should be minus.

(b) $a_4b_2c_3d_1$. (c) $a_2b_1c_4d_3$. (d) $b_4c_1d_3a_2$.
(e) $d_3b_1a_4c_2$. (f) $d_2a_1c_4b_3$. (g) $c_2a_3d_1b_4$.

2. Develop by minors the following and find the value of the determinant.

(a) $\begin{vmatrix} 3 & 2 & 4 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix}$.

Solution: $\begin{vmatrix} 3 & 2 & 4 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{vmatrix} = 3\begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix}$

$= 3(6-4) - 2(12-4) + 3(8-4) = 6 - 16 + 12 = 2.$

(b) $\begin{vmatrix} 1 & 4 & 6 \\ 7 & 8 & 2 \\ 1 & 3 & 1 \end{vmatrix}$. (c) $\begin{vmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 4 \end{vmatrix}$. (d) $\begin{vmatrix} 5 & 7 & 2 \\ 3 & 7 & 1 \\ -2 & 3 & -1 \end{vmatrix}$.

(e) $\begin{vmatrix} 0 & a & b \\ d & 0 & c \\ e & f & 0 \end{vmatrix}$. (f) $\begin{vmatrix} 0 & a & b \\ a & 0 & b \\ a & b & 0 \end{vmatrix}$. (g) $\begin{vmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{vmatrix}$.

(h) $\begin{vmatrix} 2 & 3 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 4 & 2 & 1 & 2 \\ 3 & 1 & 2 & 3 \end{vmatrix}$.

Solution: Develop with respect to the elements of the first column,

$$\begin{vmatrix} 2 & 3 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 4 & 2 & 1 & 2 \\ 3 & 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= 2\left(1\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\right) - 3\left(3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)$$

$$+ 4\left(3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right) - 3\left(3\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)$$

$$= 2(-1+2+0) - 3(-3-2+1) + 4(-3-1+1) - 3(0-1+2)$$

$$= 2 + 12 - 12 - 3 = -1.$$

(i) $\begin{vmatrix} 2 & 6 & 3 & 9 \\ 0 & 8 & 1 & 3 \\ 0 & 6 & 1 & 8 \\ 0 & 2 & 1 & 4 \end{vmatrix}$. (j) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{vmatrix}$.

HINT. It is always advisable to develop with respect to the row or column that has a maximum number of elements equal to zero.

(k) $\begin{vmatrix} a & 0 & 0 & b \\ b & a & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & b & a \end{vmatrix}$. (l) $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \\ 0 & 0 & 0 & d_4 \end{vmatrix}$.

(m) $\begin{vmatrix} 2 & 3 & 4 & 1 \\ 2 & 3 & 3 & 6 \\ 2 & 3 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{vmatrix}$. (n) $\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$.

(o) $\begin{vmatrix} x & a & b \\ b & x & a \\ a & b & x \end{vmatrix}$. (p) $\begin{vmatrix} a & 0 & g & 0 \\ e & a & f & 0 \\ 0 & 0 & a & 0 \\ d & c & b & a \end{vmatrix}$.

(q) $\begin{vmatrix} x & 0 & 0 & 0 & y \\ y & x & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 \\ 0 & 0 & y & x & 0 \\ 0 & 0 & 0 & y & x \end{vmatrix}$. (r) $\begin{vmatrix} x & a & b & c \\ c & x & a & b \\ b & c & x & a \\ a & b & c & x \end{vmatrix}$.

191. Multiplication by a constant. In this and the following sections we shall give a number of theorems on determinants which greatly facilitate their evaluation and which make a proof for the solution in terms of determinants of any number of linear equations in the same number of variables a simple matter.

THEOREM. *If every element of a row or a column is multiplied by a number m, the determinant is multiplied by m.*

Suppose that every element of the first row of a determinant is multiplied by m. Since each term of the development contains one and only one element from the first row, every term is multiplied by m, that is, the determinant is multiplied by m.

ILLUSTRATION.

$$\begin{vmatrix} ma_1 & a_2 & a_3 \\ mb_1 & b_2 & b_3 \\ mc_1 & c_2 & c_3 \end{vmatrix}$$

$$= ma_1b_2c_3 + ma_2b_3c_1 + ma_3b_1c_2 - ma_1b_3c_2 - ma_2b_1c_3 - ma_3b_2c_1$$

$$= m\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Similarly, $\begin{vmatrix} 6 & 4 & 1 \\ 8 & 3 & 2 \\ 10 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2\cdot 3 & 4 & 1 \\ 2\cdot 4 & 3 & 2 \\ 2\cdot 5 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 1 \end{vmatrix}.$

192. Interchange of rows and columns. We now prove the

THEOREM. *The value of a determinant is not changed if the columns and rows are interchanged.*

Take for instance the determinant of order four.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

In each of these determinants the principal diagonal is the same, and hence the developments derived according to the statement on

p. 209 will be the same for each determinant, since the terms will be identical except for their order. The same reasoning is valid for any determinant.

193. Interchange of rows or columns. We now prove the

THEOREM. *If two columns or two rows are interchanged, the sign of the determinant is changed.*

Again let us take for example the determinant of order four and fix our attention on the first and second rows. We must prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

In the first determinant the principal diagonal is $a_1b_2c_3d_4$, while in the second the principal diagonal, $a_2b_1c_3d_4$, is obtained from the principal diagonal of the first determinant by one inversion of subscripts. Hence this term is found among the negative terms of the first determinant.

Since the only difference between the second determinant and the first is the interchange of the subscripts 1 and 2, evidently any term of the second is obtained from some term of the first by a single inversion. Thus if a single inversion is carried out in every term of the first determinant, we obtain the various terms of the second. But since this process changes the sign of each term of the first determinant (p. 213), we see that the second determinant equals the negative of the first. Similar reasoning may be applied to the interchange of any two consecutive rows or columns of any determinant.

Consider now the effect of interchanging any two rows which are separated we will say by k intermediate rows. To bring the lower of the two rows in question to a position next below the upper one by successive interchanges of adjacent rows, we must make k such interchanges. To bring similarly the upper of the two rows to the position previously occupied by the other requires $k+1$ further interchanges of adjacent rows. Hence the interchange of the two rows is equivalent to $2k+1$ interchanges of adjacent

rows, the effect of which is to change the sign of the determinant, since $2k+1$ is always an odd number.

194. Identical rows or columns. This leads to the important

THEOREM. *If a determinant has two rows or two columns identical, its value is zero.*

Suppose that the first and the second row of a determinant are identical. Suppose that the value of the determinant is the number D. By § 193, if we interchange the first and second rows the value of the resulting determinant is $-D$. But since an interchange of two *identical* rows does not change the determinant at all, we have

$$D = -D,$$

or

$$2D = 0, \text{ that is, } D = 0.$$

COROLLARY. *If any row (or column) is m times any other row (or column), the value of the determinant is zero.*

By § 191, the determinant may be considered as the product of m and a determinant which has two rows (or columns) identical. Hence this product equals zero.

195. Proof for development by minors. On referring to the rule on p. 211 we observe that in order to show that the development by minors is the same as the development obtained by the definition on p. 209 we must prove the two following statements.

First. *The coefficient of any element in the development of a determinant (apart from sign) is the minor of that element.*

Second. *Each element times its minor should have a + or a − sign according as the sum of the number of the row and the column of the element is even or odd.*

Consider the element a_1.

First. Each term that contains a_1 must contain every other letter than a, and the indices of these letters must take on all permutations of the numbers $2, 3, \cdots, n$. This coefficient of a_1 contains then by definition (p. 210) all the terms of its minor.

Second. Since in each term a_1 is in the first place, the only inversions in the subscripts are those among the numbers $2, 3, \cdots, n$.

Hence the sign of each term in the coefficient of a_1 is positive or negative according as there is an even or odd number of inversions in its subscripts. Hence our theorem is established for the element a_1.

Consider now any element, as d_5, which occurs in the fifth row and the fourth column. Interchange adjacent rows and columns until d_5 is brought into the leading position in the principal diagonal. This requires in all seven interchanges, three to get the d_5 in the first column, and then four to get it into the first row. This changes the sign of the determinant seven times, leaving it the negative of its original value. By the reasoning just given in the case of a_1 the coefficient of d_5 (which is now in the position previously occupied by a_1) in the original determinant would be the minor of d_5, except that the signs would all be changed. Hence the term consisting of d_5 times its minor has the $-$ sign, and the theorem is proved for this case.

In general, to bring a term in the ith row and kth column to the leading position requires $i-1+k-1=i+k-2$ interchanges of adjacent rows or columns. This involves $i+k-2$, or what amounts to the same thing, $i+k$ changes of sign. Hence a positive or a negative sign should be given to an element times its minor according as $i+k$ is even or odd.

196. Sum of determinants. We now prove the fact that under certain conditions the sum of two determinants may be written in determinant form. The fact that the product of two determinants is always a determinant is extremely important for certain more advanced topics in mathematics, but the proof lies beyond the scope of this chapter.

THEOREM. *If each of the elements of any row or any column consists of the sum of two numbers, the determinant may be written as the sum of two determinants.*

For example, we have to prove that

$$\begin{vmatrix} a_1+a'_1 & b_1 & c_1 \\ a_2+a'_2 & b_2 & c_2 \\ a_3+a'_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix}.$$

Develop the first determinant by minors with respect to the first column, where we symbolize the minors of $a_1 + a'_1$, $a_2 + a'_2$, $a_3 + a'_3$ by A_1, A_2, A_3 respectively.

We have

by § 190, $\begin{vmatrix} a_1 + a'_1 & b_1 & c_1 \\ a_2 + a'_2 & b_2 & c_2 \\ a_3 + a'_3 & b_3 & c_3 \end{vmatrix}$

$$= (a_1 + a'_1)A_1 - (a_2 + a'_2)A_2 + (a_3 + a'_3)A_3,$$

by the distributive law,

$$= a_1A_1 - a_2A_2 + a_3A_3 + a'_1A_1 - a'_2A_2 + a'_3A_3,$$

by § 190, $$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix}.$$

The method of proof given is applicable to the case of any row or column of a determinant of order n.

197. Vanishing of a determinant. For the solution of systems of linear equations we shall make use of the

THEOREM. *If in the development of a determinant in terms of the minors with respect to a certain column (or row) the elements of that column (or row) are replaced by the elements of another column (or row), the resulting development equals zero.*

For example, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4, \quad (1)$$

where an A represents the minor of the a with the same subscript. We have to prove that if we replace the a's, for example, by the b's, the result equals zero, that is, that

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4 = 0. \quad (2)$$

This expression (2) when written in determinant form evidently would have the same form as the left-hand member of (1) excepting that the first column would consist of b_1, b_2, b_3, b_4. We should then have two identical columns of the determinant, which would then equal zero (§ 194). Thus the development in (2) vanishes identically. This method of proof is perfectly general.

COROLLARY. *The value of the determinant is unchanged if the elements of any row (or column) are replaced by the elements of that row (or column) increased by a multiple of the elements of another row (or column).*

Thus, for instance,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + nb_1 & b_1 & c_1 \\ a_2 + nb_2 & b_2 & c_2 \\ a_3 + nb_3 & b_3 & c_3 \end{vmatrix}.$$

The proof follows directly from §§ 191, 196, and the preceding theorem.

198. Evaluation by factoring. If in a determinant whose elements are literal two rows or two columns become identical on replacing a by b, then $a - b$ is a factor of the development. This appears immediately from § 160.

ILLUSTRATION. Evaluate by factoring

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}. \qquad (1)$$

Since two columns become identical if a is replaced by b, a by c, or b by c, then we have as a factor of the development

$$(a - b)(b - c)(c - a). \qquad (2)$$

To determine whether the signs in this product are properly chosen, that is, whether the development should contain $a - b$ or $b - a$, we note that the term bc^2 is positive in the development of (1) and also positive in the expansion of (2). Evidently there is no factor of (1) not included in (2).

199. Practical directions for evaluating determinants. In finding the value of a numerical determinant the object is to reduce it to one in which as many as possible of the elements of some row or column are zero. One should ask one's self the following questions on attempting to evaluate a determinant:

First. Is any row (or column) equal to any other row (or column)? If so, apply § 191 for the case $m = 0$.

Second. Are the elements of any row (or column) multiples of any other row (or column)? If so, apply § 191.

Third. If we add (or subtract) a multiple of the elements of one row (or column) to the elements of another, will an element be zero? If so, apply § 197.

EXERCISES

Evaluate the following determinants.

1. $\begin{vmatrix} 2 & 3 & 4 & 3 \\ 4 & 5 & 3 & 2 \\ 1 & 2 & 7 & 6 \\ 0 & 1 & 8 & 7 \end{vmatrix}$.

Solution: We observe that if we subtract each element of the first column from the corresponding element of the second column, the new second column has every element 1. A similar result is obtained by subtracting the last column from the third column. Thus, by § 191,

$$\begin{vmatrix} 2 & 3 & 4 & 3 \\ 4 & 5 & 3 & 2 \\ 1 & 2 & 7 & 6 \\ 0 & 1 & 8 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 & 3 \\ 4 & 1 & 1 & 2 \\ 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 7 \end{vmatrix} = 0.$$

2. $\begin{vmatrix} 4 & 3 & 1 & 2 \\ 6 & 1 & 1 & 3 \\ 4 & 2 & 1 & 2 \\ 3 & 6 & 2 & 1 \end{vmatrix}$.

Solution: Multiplying the last column by 2 and the whole determinant by $\frac{1}{2}$ does not change the value of the determinant (§ 191). Thus

$$\begin{vmatrix} 4 & 3 & 1 & 2 \\ 6 & 1 & 1 & 3 \\ 4 & 2 & 1 & 2 \\ 3 & 6 & 2 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 4 & 3 & 1 & 4 \\ 6 & 1 & 1 & 6 \\ 4 & 2 & 1 & 4 \\ 3 & 6 & 2 & 2 \end{vmatrix}.$$

Subtracting the last column from the first column and developing, we obtain

$$\frac{1}{2}\begin{vmatrix}4&3&1&4\\6&1&1&6\\4&2&1&4\\3&6&2&2\end{vmatrix}=\frac{1}{2}\begin{vmatrix}0&3&1&4\\0&1&1&6\\0&2&1&4\\1&6&2&2\end{vmatrix}=\frac{1}{2}\left(-1\begin{vmatrix}3&1&4\\1&1&6\\2&1&4\end{vmatrix}\right).$$

Subtracting the last row from the first row,

$$-\frac{1}{2}\begin{vmatrix}3&1&4\\1&1&6\\2&1&4\end{vmatrix}=-\frac{1}{2}\begin{vmatrix}1&0&0\\1&1&6\\2&1&4\end{vmatrix}=-\frac{1}{2}\left(1\begin{vmatrix}1&6\\1&4\end{vmatrix}\right)=-\frac{1}{2}(-2)=1.$$

3. $\begin{vmatrix}a-d&a&1\\b-d&b&1\\c-d&c&1\end{vmatrix}.$

4. $\begin{vmatrix}a^2+b^2&2ab&1\\2ab&a^2+b^2&1\\a^2&2ab+b^2&1\end{vmatrix}.$

5. $\begin{vmatrix}3&3&4&2\\1&1&2&1\\2&2&3&1\\2&1&3&2\end{vmatrix}.$

6. $\begin{vmatrix}4&6&8&3\\1&1&2&1\\2&3&4&1\\2&1&3&4\end{vmatrix}.$

7. $\begin{vmatrix}0&x&y&2\\x&0&y&2\\y&2&0&x\\2&y&x&0\end{vmatrix}.$

8. $\begin{vmatrix}1&1&0&1\\1&0&1&1\\0&1&1&1\\1&1&1&1\end{vmatrix}.$

9. $\begin{vmatrix}a&1&0&0\\b&1&1&0\\c&1&2&0\\d&1&3&3\end{vmatrix}.$

10. $\begin{vmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{vmatrix}.$

11. $\begin{vmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{vmatrix}.$

12. $\begin{vmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{vmatrix}.$

13. $\begin{vmatrix}3&7&16&14\\6&15&33&29\\0&1&1&1\\4&2&3&1\end{vmatrix}.$

14. $\begin{vmatrix}3&2&1&4\\15&29&2&14\\16&19&3&17\\33&39&8&38\end{vmatrix}.$

15. $\begin{vmatrix}9&13&17&4\\18&28&33&8\\30&40&54&13\\24&37&46&11\end{vmatrix}.$

16. $\begin{vmatrix}12&14&16&18\\2&4&6&8\\4&3&2&1\\3&7&11&15\end{vmatrix}.$

200. Solution of linear equations. Suppose that we have given n linear equations in n variables. We seek a solution of the equations in terms of determinants. For simplicity, let $n = 4$. Given

$$a_1x + b_1y + c_1z + d_1w = f_1, \qquad (1)$$
$$a_2x + b_2y + c_2z + d_2w = f_2, \qquad (2)$$
$$a_3x + b_3y + c_3z + d_3w = f_3, \qquad (3)$$
$$a_4x + b_4y + c_4z + d_4w = f_4. \qquad (4)$$

The coefficients of the variables taken in the order in which they are written may be taken as forming a determinant D, which we call the determinant of the system. Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = D.$$

Symbolize by A_1, B_3, etc., the minors of a_1, b_3, etc., in this determinant. Let us solve for x. Multiply (1), (2), (3), (4) by A_1, A_2, A_3, A_4 respectively. We obtain

$$A_1a_1x + A_1b_1y + A_1c_1z + A_1d_1w = A_1f_1,$$
$$A_2a_2x + A_2b_2y + A_2c_2z + A_2d_2w = A_2f_2,$$
$$A_3a_3x + A_3b_3y + A_3c_3z + A_3d_3w = A_3f_3,$$
$$A_4a_4x + A_4b_4y + A_4c_4z + A_4d_4w = A_4f_4.$$

If we add these equations, having changed the signs of the second and fourth, the coefficient of x is the determinant D, while the coefficients of y, z, w are zero (§ 197). The right-hand member of the equation is the determinant D, excepting that the elements of the first column are replaced by f_1, f_2, f_3, f_4 respectively. Hence

$$x = \frac{\begin{vmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ f_3 & b_3 & c_3 & d_3 \\ f_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}}.$$

In a similar manner we can show that the value of any variable which satisfies the equations is given by the following

RULE. *The value of one of the variables in the solution of n linear equations in n variables consists of a fraction whose denominator is the determinant of the system and whose numerator is the same determinant, excepting that the column which contains the coefficients of the given variable is replaced by a column consisting of the constant terms.*

When $D = 0$, we cannot solve the equations unless the numerators in the expressions for the solution also vanish.

ILLUSTRATION. Solve for x

$$\begin{aligned} ax + 2by &= 1, \\ 2by + 3cz &= 2, \\ 3cz + 4dw &= 3, \\ 4dw + 5ax &= 4. \end{aligned}$$

Rearranging, we obtain

$$\begin{array}{lllll} ax & + 2by & & & = 1, \\ & 2by & + 3cz & & = 2, \\ & & 3cz & + 4dw & = 3, \\ 5ax & & & + 4dw & = 4. \end{array}$$

$$D = \begin{vmatrix} a & 2b & 0 & 0 \\ 0 & 2b & 3c & 0 \\ 0 & 0 & 3c & 4d \\ 5a & 0 & 0 & 4d \end{vmatrix} = 24 \begin{vmatrix} a & b & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & c & d \\ 5a & 0 & 0 & d \end{vmatrix}$$

$$x = \frac{24\begin{vmatrix} 1 & b & 0 & 0 \\ 2 & b & c & 0 \\ 3 & 0 & c & d \\ 4 & 0 & 0 & d \end{vmatrix}}{24\begin{vmatrix} a & b & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & c & d \\ 5a & 0 & 0 & d \end{vmatrix}} = \frac{\begin{vmatrix} 1 & b & 0 & 0 \\ 1 & 0 & c & 0 \\ 3 & 0 & c & d \\ 4 & 0 & 0 & d \end{vmatrix}}{\begin{vmatrix} a & b & 0 & 0 \\ 0 & b & c & 0 \\ 0 & 0 & c & d \\ 0 & -5b & 0 & d \end{vmatrix}} = \frac{-b\begin{vmatrix} 1 & c & 0 \\ 3 & c & 0 \\ 4 & 0 & d \end{vmatrix}}{a\begin{vmatrix} b & c & 0 \\ 0 & c & d \\ -5b & 0 & d \end{vmatrix}}$$

$$= \frac{-b\begin{vmatrix} 1 & c & 0 \\ 2 & 0 & 0 \\ 4 & 0 & d \end{vmatrix}}{a\begin{vmatrix} b & c & 0 \\ 0 & c & d \\ 0 & 5c & d \end{vmatrix}} = \frac{bc\begin{vmatrix} 2 & 0 \\ 4 & d \end{vmatrix}}{ab\begin{vmatrix} c & d \\ 5c & d \end{vmatrix}} = \frac{2\,bcd}{-4\,abcd} = -\frac{1}{2\,a}.$$

201. Solution of homogeneous linear equations. The equations considered in the previous section become homogeneous (p. 115) if $f_1 = f_2 = f_3 = f_4 = 0$. We have then

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= 0, \\ a_2x + b_2y + c_2z + d_2w &= 0, \\ a_3x + b_3y + c_3z + d_3w &= 0, \\ a_4x + b_4y + c_4z + d_4w &= 0. \end{aligned} \tag{I}$$

These equations have evidently the solution $x = y = z = w = 0$. This we call the zero solution. We seek the condition that the coefficients must fulfill in order that other solutions also may exist. If we carry out the method of the previous section, we observe that the determinant equals zero in the numerator of every fraction which affords the value of one of the variables (§ 191). Thus if D is not equal to zero, the only solution of the above equations is the zero solution. This gives us the following

PRINCIPLE. *A system of n linear homogeneous equations in n variables has a solution distinct from the zero solution only when the determinant of the system vanishes.*

Whether a solution distinct from the zero solution *always* exists when the determinant of the system equals zero we shall not determine, as a complete discussion of the question would be beyond the scope of this chapter.

THEOREM. *If x_1, y_1, z_1, w_1 is a solution of equations* (I) *and k is any number, then kx_1, ky_1, kz_1, kw_1 is also a solution.*

The proof of this theorem is evident on substituting kx_1, etc., in equations (I) and observing that the number k is a factor of each equation. Thus if a system of n linear homogeneous equations has any solution distinct from the zero solution it has an infinite number of solutions.

EXERCISES

Solve for all the variables:

1. $$\begin{aligned} x+y&=a,\\ y+z&=b,\\ z+u&=c,\\ u+v&=d,\\ v+x&=e. \end{aligned}$$

2. $$\begin{aligned} x+3y&=19,\\ y+3z&=8,\\ z+3u&=7,\\ u+3v&=11,\\ v+3x&=15. \end{aligned}$$

3. $$\begin{aligned} z+y+w&=a,\\ z+w+x&=b,\\ w+x+y&=c,\\ x+y+z&=d. \end{aligned}$$

4. $$\begin{aligned} 3x+y+z&=20,\\ x+4y+3w&=30,\\ 6x+z+3w&=40,\\ 8y+3z+5w&=50. \end{aligned}$$

5. $$\begin{aligned} y+z+5w&=11,\\ x+z+4w&=11,\\ x+y+3w&=11,\\ x+z+8y&=33. \end{aligned}$$

6. $$\begin{aligned} x-2y+3z-w&=5,\\ y-2z+3w-x&=0,\\ z-2w+3x-y&=0,\\ w-2x+3y-z&=5. \end{aligned}$$

7. $$\begin{aligned} x+y+z+w&=24,\\ x+2y+3z-9w&=0,\\ 3x-y-5z+w&=0,\\ 2x+3y-4z-5w&=0. \end{aligned}$$

8. $$\begin{aligned} x+y+z+w&=60,\\ x+2y+3z+4w&=100,\\ x+3y+6z+10w&=150,\\ x+4y+10z+20w&=210. \end{aligned}$$

CHAPTER XIX

PARTIAL FRACTIONS

202. Introduction. For various purposes it is convenient to express a rational algebraic expression $\frac{f(x)}{\phi(x)}$, § 11, as the sum of several fractions called **partial fractions**, which have the several factors of $\phi(x)$ as denominators and which have constants for numerators. If we write $\phi(x) = (x-\alpha)(x-\beta)\cdots(x-\nu)$, we seek a means of determining constants $A, B, \cdots, N$ such that for every value of x

$$\frac{f(x)}{\phi(x)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \cdots + \frac{N}{x-\nu}. \tag{1}$$

If the degree of $f(x)$ is equal to or greater than that of $\phi(x)$, we can write

$$\frac{f(x)}{\phi(x)} = Q(x) + \frac{f_1(x)}{\phi(x)}, \tag{2}$$

where $Q(x)$ is the quotient and $f_1(x)$ the remainder from dividing $f(x)$ by $\phi(x)$, and where the degree of $f_1(x)$ is less than that of $\phi(x)$. In what follows we shall assume that the degree of $f(x)$ is less than that of $\phi(x)$. In problems where this is not the case one should carry out the long division indicated by (2) and apply the principle developed in this chapter to the expression corresponding to the last term in (2).

203. Development when $\phi(x) = 0$ has no multiple roots. Let us consider the particular case

$$\frac{f(x)}{\phi(x)} = \frac{x+1}{(x-1)(x-2)(x-3)}.$$

We indicate the development required in form (1) of the last section,

$$\frac{x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}, \tag{1}$$

where A, B, and C represent constants which we are to determine if possible. The question arises immediately, Are we at liberty to make this assumption? Are we not assuming the essence of what we wish to prove, i.e. the form of the expansion? To this we may answer, We have written the expansion in form (1) tentatively. We have not proved it and are not certain of its validity. If, however, we are able to find numerical values of A, B, and C which satisfy (1), we can then write down the actual development of the fraction in the form of an identity.

If, on the other hand, we can show that no such numbers A, B, C satisfying (1) exist, then the development is not possible.

Clear (1) of fractions,

$$\begin{aligned} x+1 &= A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \\ &= (A+B+C)x^2-(5A+4B+3C)x+6A+3B+2C. \end{aligned}$$

Since we seek values of A, B, and C for which (1) is identically true for all values of x, equate coefficients of like powers of x in the last equation (Corollary II, p. 174). We obtain

$$A+B+C=0, \tag{2}$$

$$-5A-4B-3C=1, \tag{3}$$

$$6A+3B+2C=1. \tag{4}$$

Add (4) to (3) and we obtain

$$A-B-C=2, \tag{5}$$

$$A+B+C=0 \tag{2}$$

Adding we obtain $\quad 2A=2$

$$A=1$$

From (3) and (4) we obtain

$$B=-3,\ C=2.$$

Thus $$\frac{x+1}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}-\frac{3}{x-2}+\frac{2}{x-3}.$$

As a check we might clear of fractions and simplify. If equations (2), (3), and (4) had been incompatible, we should have concluded that we could not develop the fraction in form (1).

We assume now for the general case

$$\phi(x) = (x - \alpha)(x - \beta)\cdots(x - \nu),$$

and that the roots $\alpha, \beta, \cdots, \nu$ are all distinct from each other.

Let us consider the expression

$$\frac{f(x)}{\phi(x)} = \frac{A}{x - \alpha} + \frac{B}{x - \beta} + \cdots + \frac{N}{x - \nu}, \tag{6}$$

where $A, B, \cdots, N$ are constants. Let us assume for the moment the possibility of expressing $\frac{f(x)}{\phi(x)}$ in terms of these partial fractions. We shall now attempt to determine actual values $A, B, \cdots, N$ which satisfy such an identity. If we multiply both sides of the identity by

$$\phi(x) = (x - \alpha)(x - \beta)\cdots(x - \nu),$$

we obtain

$$f(x) = A(x - \beta)\cdots(x - \nu) + B(x - \alpha)\cdots(x - \nu) + \cdots \\ + N(x - \alpha)(x - \beta)\cdots.$$

$f(x)$ is of degree not greater than $n - 1$, and consequently when written in the form of (1), p. 166, has not more than n terms. If we multiply out the right-hand member and collect powers of x, we have an expression in x of degree $n - 1$. By Corollary II, p. 174, this expression will be an identity if we can determine values of $A, B, \cdots, N$ which make the coefficients of x on both sides of the equation equal to each other. Hence we equate coefficients of like powers of x and obtain n equations linear in $A, B, \cdots, N$ which we can treat as variables. These equations have in general one and only one solution which we can easily determine. The values of $A, B, \cdots, N$ obtained by solving these equations we can substitute for the numerators of the partial fractions in (6). After making this substitution we can actually clear of fractions the right-hand member of (6) and check our work by showing its identity with the left-hand member.

There is no general criterion that we have applied to (6) to determine whether the n linear equations obtained by equating coefficients of x have any solution or not. Hence in this general

discussion it should be distinctly understood that assumption (6) holds when and only when these equations are solvable. In any particular case we can find out immediately whether the equations are solvable by attempting to solve them. If the numbers $A, B, \cdots, N$ do not exist, the fact will appear by our inability to solve the linear equations. As a matter of fact, one and only one solution always exists under the assumption of this section.

If in (6) we assume that several of the symbols $A, B, \cdots, N$ stand for expressions linear in x, as, for instance, $ax + b$, we should then have a larger number of variables to determine than there are equations. Under these circumstances there is an infinite number of solutions of the equations. Thus if we should seek to express $\frac{f(x)}{\phi(x)}$ as the sum of partial fractions where the numerators are not constants but functions of x, we could get any number of such developments.

We have the following

RULE. *Factor $\phi(x)$ into linear factors, as*

$$(x-\alpha)(x-\beta)\cdots(x-\nu).$$

Write the expression

$$\frac{f(x)}{\phi(x)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \cdots + \frac{N}{x-\nu}.$$

Multiply both sides of the expression by $\phi(x)$, equate coefficients of like powers of x, and solve the resulting linear equations for $A, B, \cdots, N$.

Replace $A, B, \cdots, N$ by these values and check by substituting for x some number distinct from $\alpha, \beta, \cdots, \nu$.

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2-2}{(x-1)(x-2)x}$.

Solution: Assume $$\frac{x^2-2}{(x-1)(x-2)x} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x}. \qquad (1)$$

Multiply by $(x-1)(x-2)x$,

$$x^2 - 2 = A(x-2)x + B(x-1)x + C(x-1)(x-2),$$
$$x^2 - 2 = (A + B + C)x^2 - (2A + B + 3C)x + 2C.$$

Equating coefficients of like powers of x,

$$A + B + C = 1,$$
$$2A + B + 3C = 0,$$
$$2C = -2.$$

Hence $C = -1.$

Solving
$$A + B = 2,$$
$$2A + B = 3,$$
we obtain
$$A = 1$$
$$B = 1$$

Thus
$$\frac{x^2 - 2}{(x-1)(x-2)x} = \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x}.$$

Check: Let $x = -1,$

Substituting in (1),
$$\frac{-1}{-6} = \frac{1}{-2} + \frac{1}{-3} - \frac{1}{-1},$$
or
$$\frac{1}{6} = -\frac{1}{2} - \frac{1}{3} + 1 = -\frac{5}{6} + 1 = \frac{1}{6}.$$

2. $\dfrac{x-1}{x^2 + 3x + 2}.$

3. $\dfrac{x+7}{2x^2 - 5x - 3}.$

4. $\dfrac{1}{3x^2 - 2x - 8}.$

5. $\dfrac{5x}{6x^2 - 5x - 1}.$

6. $\dfrac{4x^2}{(x^2 - 4)(x-1)}.$

7. $\dfrac{2x^2 - 1}{(x^2 + 3x + 2)(x-1)}.$

8. $\dfrac{x^2 - 3x + 1}{(x-1)(x-2)(x-3)}.$

9. $\dfrac{x^2 + 4}{(x-2)(x+2)(x-1)}.$

204. Development when $\phi(x) = 0$ has imaginary roots. In the preceding section no mention has been made of any distinction between real and imaginary values of $\alpha, \beta, \cdots, \nu$. In fact the method given is valid whether they are real or imaginary. It is, however, desirable to obtain a development in which only real numbers appear.

Let us assume the development

$$\frac{f(x)}{\phi(x)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \cdots + \frac{m}{x-\mu} + \frac{n}{x-\nu}, \qquad (1)$$

where let us suppose that μ and ν are *the only* pair of conjugate imaginary roots of $\phi(x)$, m and n being conjugate complex numbers.

Let $$\mu = a + ib,\ \nu = a - ib.$$

Then adding the corresponding terms of (1), we obtain

$$\frac{m}{x-a-ib}+\frac{n}{x-a+ib}=\frac{x(m+n)-a(m+n)}{(x-a)^2+b^2}+i\frac{b(m-n)}{(x-a)^2+b^2}. \qquad (2)$$

Since μ and ν are the only imaginary roots of $\phi(x)=0$, the last term of (2) is real, as is also the entire right-hand member (§ 152). Hence, letting the numerator

$$x(m+n)-a(m+n)+ib(m-n)=Mx+N,$$

we have the development

$$\frac{f(x)}{\phi(x)}=\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\cdots+\frac{Mx+N}{(x-a)^2+b^2}. \qquad (3)$$

By complete induction we can establish this form of numerator where there is any number of pairs of imaginary roots of $\phi(x)=0$. We have proved the form (3) where there is one pair of imaginary roots. Assuming the form where there are k pairs, we can prove it similarly where there are $k+1$ pairs. Hence we have the

THEOREM. *If $\phi(x)$ is factorable into distinct linear and quadratic factors, but the quadratic factors are not further reducible into real * factors, then $\dfrac{f(x)}{\phi(x)}$ is separable into partial fractions of the form*

$$\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\cdots+\frac{Mx+N}{x^2+\mu x+\nu},$$

where $x^2+\mu x+\nu$ is an irreducible quadratic factor of $\phi(x)$.

This theorem is of course true only under the condition that the linear equations obtained in the process of determining the constants are solvable. It turns out, however, that in this case as in § 203 the linear equations obtained always have one and only one solution provided that the roots are all distinct.

* A *real factor* is one whose coefficients are all real numbers.

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2+1}{(x-1)(x^2+x+1)}.$

Solution: Assume $\dfrac{x^2+1}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}.$

Multiply by $(x-1)(x^2+x+1)$,

$$x^2+1 = A(x^2+x+1) + (Bx+C)(x-1).$$

Collecting like powers of x,

$$x^2+1 = (A+B)x^2 + (A-B+C)x + A - C.$$

Equating coefficients of x,

$$A + B = 1, \quad (1)$$
$$A - B + C = 0, \quad (2)$$
$$A - C = 1. \quad (3)$$

Add (2) and (3) and solve with (1),

$$\begin{array}{r} 2A - B = 1 \\ \underline{A + B = 1} \\ 3A = 2, \end{array}$$

or $\quad A = \frac{2}{3}.$

Substituting in (1), $\quad \frac{2}{3} + B = 1,$

or $\quad B = \frac{1}{3}.$

Substituting in (3), $\quad \frac{2}{3} - C = 1,$

or $\quad C = -\frac{1}{3}.$

Thus $\dfrac{x^2+1}{(x-1)(x^2+x+1)} = \dfrac{2}{3(x-1)} + \dfrac{x-1}{3x^2+3x+3}.$

Check: Let $x = -1$.

Substituting, $\quad \dfrac{2}{-2\cdot 1} = \dfrac{2}{3\cdot -2} + \dfrac{-2}{3-3+3}.$

Reducing, $\quad -1 = -\dfrac{2}{6} - \dfrac{2}{3} = -1.$

2. $\dfrac{x^2+x+1}{x^3+4x}.$

3. $\dfrac{x^2+1}{x^4+x^2+1}.$

4. $\dfrac{x^2+4}{x^3-2x^2+3x-2}.$

5. $\dfrac{5x^3-1}{x^4+6x^2+8}.$

6. $\dfrac{x}{(x+3)(2x^2-x-4)}.$

7. $\dfrac{1}{x^3+3x^2-2x-16}.$

HINT. Factor by synthetic division (see p. 178).

8. $\dfrac{x^2+1}{(x-1)(3x^2+x+6)}.$

205. Development when $\phi(x)=(x-a)^n$. In this case the method given in the previous sections fails, as the equations for determining the values of the numerators are incompatible. If we let

$$\frac{f(x)}{\phi(x)}=\frac{a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1}}{(x-a)^n}, \tag{1}$$

we can separate into partial fractions as follows.

Let $x-a=y$, that is, $x=y+a$, and substitute in (1). We obtain after collecting powers of y

$$\frac{A_0y^{n-1}+A_1y^{n-2}+\cdots+A_{n-1}}{y^n}=\frac{A_0}{y}+\frac{A_1}{y^2}+\cdots+\frac{A_{n-1}}{y^n},$$

where the A's are constants. Replacing y by $x-a$, we have the following development:

$$\frac{f(x)}{(x-a)^n}=\frac{A_0}{x-a}+\frac{A_1}{(x-a)^2}+\frac{A_2}{(x-a)^3}+\cdots+\frac{A_{n-1}}{(x-a)^n}.$$

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2+2x-1}{(x-3)^3}$.

Solution: Assume $\dfrac{x^2+2x-1}{(x-3)^3}=\dfrac{A}{x-3}+\dfrac{B}{(x-3)^2}+\dfrac{C}{(x-3)^3}$. (1)

Multiply by $(x-3)^2$,

$$x^2+2x-1=A(x^2-6x+9)+B(x-3)+C.$$

Collecting powers of x,

$$=Ax^2+(B-6A)x+9A-3B+C.$$

Equating coefficients of x,

$$A=1,$$
$$B-6A=2,$$
$$9A-3B+C=-1.$$

Solving, $B=8,\ C=14.$

Hence $\dfrac{x^2+2x-1}{(x-3)^3}=\dfrac{1}{x-3}+\dfrac{8}{(x-3)^2}+\dfrac{14}{(x-3)^3}$.

Check: Let $x=1$.

Substituting in (1), $\dfrac{2}{-8}=\dfrac{1}{-2}+\dfrac{8}{4}-\dfrac{14}{8}=-\dfrac{1}{2}+2-\dfrac{14}{8}$,

or $-\dfrac{1}{4}=-\dfrac{1}{4}$.

2. $\dfrac{x^2-4}{(x-2)^3}$. 3. $\dfrac{x}{(2x+1)^2}$. 4. $\dfrac{x}{(x-4)^3}$.

5. $\dfrac{x^2+x+1}{(2x-1)^4}$. 6. $\dfrac{x-a}{(ax+b)^2}$. 7. $\dfrac{2x^2+3x+1}{(3x-2)^3}$.

206. General case. When $\phi(x)=0$ has real, complex, and multiple roots, we may use all the previous methods simultaneously. Hence for this case we assume the expansion

$$\frac{f(x)}{(x-\alpha)\cdots(\lambda x+\mu x+\nu)\cdots(x-\tau)^k}$$
$$=\frac{A}{x-\alpha}+\cdots+\frac{Mx+N}{\lambda x+\mu x+\nu}+\cdots+\frac{T}{x-\tau}+\cdots+\frac{V}{(x-\tau)^k}.$$

EXERCISES

Separate into partial functions:

1. $\dfrac{x^4+2x^2+18x-18}{(x-1)(x^2+x+1)(x-2)^2}$.

Solution:

$$\frac{x^4+2x^2+18x-18}{(x-1)(x^2+x+1)(x-2)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}+\frac{D}{x-2}+\frac{E}{(x-2)^2}$$
$$=A(x^2+x+1)(x-2)^2+(Bx+C)(x-2)^2(x-1)$$
$$+D(x-1)(x-2)(x^2+x+1)$$
$$+E(x-1)(x^2+x+1)$$
$$=(A+B+D)x^4+(-3A-5B+C-2D+E)x^3$$
$$+(A+8B-5C)x^2+(-4B+8C-D)x$$
$$+(4A-4C+2D-E).$$

Equating coefficients of like powers of x,

$$A+B+D=1, \tag{1}$$
$$-3A-5B+C-2D+E=0, \tag{2}$$
$$A+8B-5C=2, \tag{3}$$
$$-4B+8C-D=18, \tag{4}$$
$$4A-4C+2D-E=-18. \tag{5}$$

Adding (2) and (5), (1) and (4), we have, together with (3),

$$A-5B-3C=-18, \tag{6}$$
$$A-3B+8C=19, \tag{7}$$
$$A+8B-5C=2. \tag{3}$$

Adding all three equations, we find

$$3A = 3, \text{ or } A = 1.$$

Substituting in (3) and (7) and solving, we find $C = 3$, $B = 2$. Substituting in (1), we find $D = -2$. Similarly, from (5), $E = 6$.

Thus $$\frac{x^4 + 2x^3 + 18x - 18}{(x-1)(x^2+x+1)(x-2)^2} = \frac{1}{x-1} + \frac{2x+3}{x^2+x+1} - \frac{2}{x-2} + \frac{6}{(x-2)^2}.$$

Check: Let $x = -1$.

Substituting, $$\frac{-33}{-2 \cdot 1 \cdot 9} = \frac{1}{-2} + \frac{1}{1} - \frac{2}{-3} + \frac{6}{9},$$

or $$\tfrac{11}{6} = 2\tfrac{1}{3} - \tfrac{1}{2} = \tfrac{11}{6}.$$

2. $\dfrac{x^3+1}{x(x-1)^3}$.

3. $\dfrac{4x^2}{1-x^4}$.

4. $\dfrac{x^3+1}{x(x-1)^3}$.

5. $\dfrac{5x+12}{x(x^2+4)}$.

6. $\dfrac{2x^3+x+3}{(x^2+1)^2}$.

7. $\dfrac{43x-11}{30(x^2-1)}$.

8. $\dfrac{x-1}{(x+1)^2(x+2)}$.

9. $\dfrac{x^3-x-1}{x^4-16}$.

10. $\dfrac{11x+7}{(x^2-1)(x+2)}$.

11. $\dfrac{x^2+6x-8}{x^3-4x}$.

12. $\dfrac{x^2-6x+1}{(x-8)(x-9)}$.

13. $\dfrac{2}{(x^2+x+3)(2x+1)}$.

14. $\dfrac{2x^2-3x-3}{(x-1)(x^2-2x+5)}$.

15. $\dfrac{x^3+2x^2-3x+1}{(x+3)(x^2-4x+5)}$.

16. $\dfrac{30x-17}{(2x-3)(6x^2-5x+1)}$.

17. $\dfrac{13-5x}{(x^2-3x+2)(x-3)}$.

CHAPTER XX

LOGARITHMS

207. Generalized powers. If b and c are integers, we can easily compute b^c. When c is not an integer but a fraction we can compute the value of b^c to any desired degree of accuracy. Thus if $b=2$, $c=\frac{3}{2}$, we have $2^{\frac{3}{2}}=\sqrt{2^3}=\sqrt{8}$, which we can find to any number of decimal places. If, however, the exponent is an irrational number as $\sqrt{2}$, we have shown no method of computing the expression. Since, however, $\sqrt{2}$ was seen (p. 55) to be the limit approached by the sequence of numbers

$$1,\ 1.4,\ 1.41,\ 1.414,\ \cdots,$$

it turns out that $5^{\sqrt{2}}$ is the limit approached by the numbers

$$5^{1.},\ 5^{1.4},\ 5^{1.41},\ 5^{1.414},\ \cdots.$$

The computation of such a number as $5^{1.41}$ would be somewhat laborious, but could be performed, since $5^{1.41}=5^{\frac{141}{100}}=\sqrt[100]{5^{141}}$. Thus it is a root of the equation $x^{100}=5^{141}$, and could be found by Horner's method, p. 197.

We see in this particular case that $5^{\sqrt{2}}$ is the limit approached by a sequence of numbers where the exponents are the successive approximations to $\sqrt{2}$ obtained by the process of extracting the square root. In a similar manner we could express the meaning of b^c, where b is a positive integer and c is any irrational number.

ASSUMPTION. *We assume that the laws of operation which we have adopted for rational exponents hold when the exponents are irrational.*

Thus $\qquad b^c\cdot b^d=b^{c+d},\ \dfrac{b^c}{b^d}=b^{c-d},\ (b^d)^c=(b^c)^d=b^{cd},$

where c and d are any numbers, rational or irrational.

208. Logarithms. We have just seen that when b and c are given a number a exists such that $b^c = a$. We now consider the case where a and b are given and c remains to be found. Let $a = 8$, $b = 2$. Then if $2^c = 8$, we see immediately that $c = 3$ satisfies this equation. If $a = 16$, $b = 2$, then $2^c = 16$ and $c = 4$ is the solution. If $a = 10$, $b = 2$, consider the equation $2^c = 10$. If we let $c = 3$, we see that $2^3 = 8$. If we let c equal the next larger integer, 4, we see $2^4 = 16$. If then any number c exists such that $2^c = 10$, it must evidently lie between 3 and 4. To prove the existence of such a number is beyond the scope of this chapter, but we make the following

ASSUMPTION. *There always exists a real number x which satisfies the equation*

$$b^x = a, \tag{1}$$

where a and b are positive numbers, provided $b \neq 1$.

Since any real number is expressible approximately in terms of a decimal fraction, this number x is so expressible.

The power to which a given number called the base must be raised to equal a second number is called the logarithm of the second number.

In (1) x is the logarithm of a for the base b.

This is abbreviated into

$$x = \log_b a. \tag{2}$$

Thus since

$$2^3 = 8,\ 10^2 = 100,\ 3^{-2} = \tfrac{1}{9},\ 4^0 = 1,$$

we have

$$3 = \log_2 8,\ 2 = \log_{10} 100,\ -2 = \log_3 \tfrac{1}{9},\ 0 = \log_4 1.$$

The number a in (1) and (2) is called the antilogarithm.

EXERCISES

1. In the following name the base, the logarithm, and the antilogarithm, and write in form (2).

(a) $3^6 = 739$.

Solution: $3 =$ base, $6 =$ logarithm, $739 =$ antilogarithm, $\log_3 739 = 6$.

(b) $2^4 = 16$. (c) $3^3 = 27$.

2. Find the logarithms of the following numbers for the base 3:

$$81,\ 243,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{81}.$$

3. For base 2 find logarithms of 8, 128, $\frac{1}{2}$, $\frac{1}{16}$.

4. What must the base be when the following equations are true?

(a) $\log 49 = 2$. (b) $\log 81 = 4$.

(c) $\log 225 = 2$. (d) $\log 625 = 4$.

209. Operations on logarithms. By means of the law expressed in the Assumption, § 207, we arrive at principles that have made the use of logarithms the most helpful aid in computations that is known.

THEOREM I. *The logarithm of the product of two numbers is the sum of their logarithms.*

Let $\log_b a = x,$

$\log_b c = y.$

Then by (1) and (2), p. 236, $b^x = a,$

$b^y = c.$

Multiply (by Assumption, § 207),

$$b^{x+y} = a \cdot c,$$

or by (1) and (2), $\log a \cdot c = x + y.$

THEOREM II. *The logarithm of the nth power of a number is n times the logarithm of the number.*

Let $\log_b a = x,$

or $b^x = a.$

Raise both sides to the nth power,

$$(b^x)^n = b^{nx} = a^n,$$

or $\log_b a^n = nx.$

EXAMPLE. $\log_{10} 100 = 2,$

$\log_{10} 1000 = 3.$

By Theorem I, $\log_{10} 100{,}000 = 5,$

which is evidently true, since $10^5 = 100{,}000.$

THEOREM III. *The logarithm of the quotient of two numbers is the difference between the logarithms of the numbers.*

Let $$\log_b a = x,$$
$$\log_b c = y.$$

Then $$b^x = a,$$
$$b^y = c.$$

Dividing, $$b^{x-y} = \frac{a}{c},$$

or $$\log \frac{a}{c} = x - y.$$

THEOREM IV. *The logarithm of the real nth root of a number is the logarithm of the number divided by n.*

Let $$\log_b a = x,$$

or $$b^x = a.$$

Extract the nth root, $$(b^x)^{\frac{1}{n}} = b^{\frac{x}{n}} = \sqrt[n]{a},$$

or $$\log_b \sqrt[n]{a} = \frac{x}{n}.$$

EXERCISES

Given $\log_{10} 2 = .301$, $\log_{10} 5 = .699$, $\log_{10} 7 = .8451$, find

1. $\log(\sqrt[5]{7^3} \cdot \sqrt{5})$.*

Solution:

By Theorem I, $\log(\sqrt[5]{7^3} \cdot \sqrt{5}) = \log \sqrt[5]{7^3} + \log \sqrt{5}$.

By Theorems III and IV, $= \frac{3}{5} \log 7 + \frac{1}{2} \log 5$.

Now $\frac{3}{5} \log 7 = \frac{3}{5}(.8451) = .50706$,

and $\frac{1}{2} \log 5 = \frac{1}{2}(.699) = .3495$

Adding, $\frac{3}{5} \log 7 + \frac{1}{2} \log 5 = \log(\sqrt[5]{7^3} \cdot \sqrt{5}) = .85656$

2. $\log 40$.

HINT. Let $40 = 8 \cdot 5 = 2^3 \cdot 5$.

3. $\log 28$.

4. $\log 140$.

5. $\log \sqrt{280}$.

6. $\log \sqrt[3]{35}$.

7. $\log(\sqrt{8} \cdot \sqrt[3]{5^2} \cdot \sqrt[5]{7})$.

8. $\log(\sqrt{5} \cdot 7^3)$.

9. $\log(\sqrt[3]{16} \cdot \sqrt{14} \cdot \sqrt[4]{700})$.

* Where no base is written it is assumed that the base 10 is employed.

210. Common system of logarithms. For purposes of computation 10 is taken as a base, and unless some other base is indicated we shall assume that such is the case for the rest of this chapter. We may write as follows the equations which show the numbers of which integers are the logarithms.

Since	$10^5 = 100{,}000$ we have	$\log 100{,}000 = 5.$
	$10^4 = 10{,}000$	$\log 10{,}000 = 4.$
	$10^3 = 1000$	$\log 1000 = 3.$
	$10^2 = 100$	$\log 100 = 2.$
	$10^1 = 10$	$\log 10 = 1.$
	$10^0 = 1$	$\log 1 = 0.$
	$10^{-1} = .1$	$\log .1 = -1.$
	$10^{-2} = .01$	$\log .01 = -2.$
	$10^{-3} = .001$	$\log .001 = -3.$
	etc.	etc.

Assuming that as x becomes greater $\log x$ also becomes greater, we see that a number, for example, between 10 and 100 has a logarithm between 1 and 2. In fact the logarithm of any number not an exact power of 10 consists of a whole-number part and a decimal part.

Thus since $10^3 < 3421 < 10^4$,

$$\log 3421 = 3. + \text{a decimal}.$$

Since $10^{-3} < .0023 < 10^{-2}$,

$$\log .0023 = -3. + \text{a decimal}.$$

The whole-number part of the logarithm of a number is called the **characteristic** of the logarithm.

The decimal part of the logarithm of a number is called the **mantissa** of the logarithm.

The characteristic of the logarithm of any number may be seen from the above table, from which the following rules are immediately deduced.

The characteristic of the logarithm of a number greater than unity is one less than the number of digits to the left of its decimal point.

Thus the characteristic of the logarithm of 471 is 2, since 471 is between 100 and 1000; of 27.93 is 1, since this number is between 10 and 100; of 8964.2 is 3, since this number is between 1000 and 10,000.

The characteristic of the logarithm of a number less than 1 is one greater negatively than the number of zeros preceding the first significant figure.

Thus the characteristic of the logarithm of .04 is -2; of .006791 is -3; of .4791 is -1.

It must constantly be kept in mind that the logarithm of a number less than 1 consists of a negative integer as a characteristic *plus* a positive mantissa. To avoid complication it is desirable always to add 10 to and subtract 10 from a logarithm when the characteristic is negative. Thus, for instance, instead of writing the logarithm $-3+.4672$ we write $10-3+.4672-10$, or $7.4672-10$. This is convenient when for example we wish to divide a logarithm by 2, as by Theorem IV, § 209, we shall wish to do when we extract a square root. Since in the logarithm $-3+.4672$ the mantissa is positive, it would not be correct to divide -3.4672 by 2, as we should confuse the positive and negative parts. This confusion is avoided if we use the form $7.4672-10$, and the result of division by 2 is $3.7336-5$, or $8.7336-10$. The actual logarithm which is the result of this division is $-2+.7336$.

THEOREM. *Numbers with the same significant figures which differ only in the position of their decimal points have the same mantissa.*

Consider for example the numbers 24.31 and 2431.

Let $$10^x = 24.31.$$

Then $$x = \log 24.31.$$

If we multiply both numbers of this equation by 100, we have

$$10^2 10^x = 10^{x+2} = 2431,$$

or $$x + 2 = \log 2431.$$

Thus the logarithm of one number differs from that of the other merely in the characteristic. In general numbers with the same significant figures are identical except for multiples of 10. Hence their logarithms differ only by integers, leaving their mantissas the same.

Thus if $\log 47120. = 4.6732$, $\log 47.12 = 1.6732$, and $\log .004712 = -3.6732$, or $7.6732-10$.

EXERCISES

If $\log 2 = .3010$, $\log 3 = .4771$, $\log 7 = .8451$, find

1. $\log \sqrt{600}$.

Solution: $\log \sqrt{600} = \log \sqrt{20 \cdot 30} = \frac{1}{2} \log 20 + \frac{1}{2} \log 30$.

By the preceding theorem, $\log 20 = 1.3010$, $\log 30 = 1.4771$.

$$\frac{1}{2} \log 20 = .6505$$
$$\frac{1}{2} \log 30 = .73855$$

By § 209, $\log \sqrt{600} = 1.38905$

2. $\log .06$. **3.** $\log (210)^3$. **4.** $\log 5.4$.

5. $\log (4.2)^4$. **6.** $\log \frac{9.6}{2.1}$. **7.** $\log \frac{(70)^3}{324}$.

8. $\log \frac{567}{1.6}$. **9.** $\log \frac{13.23}{1.28}$.

211. Use of tables. A table of logarithms contains the mantissas of the logarithms of all numbers of a certain number of significant figures. The table found later in this chapter gives immediately the mantissas for all numbers of three significant figures. In the next section a method is given for finding the mantissa for a number of four figures. Hence the table is called a four-place table. Before every mantissa in the table a decimal point is assumed to stand, but in order to save space it is not written. To find the logarithm of a number of three or fewer significant figures we apply the following

RULE. *Determine the characteristic by rules in* § 210.

Find in column N the first two significant figures of the number. The mantissa required is in the row with these figures.

Find at the top of the page the last figure of the number. The mantissa required is in the column with this figure.

When the first significant figure is 1 we may find the logarithm of any number of four figures by this rule from the table on pp. 248, 249 if we find the first three instead of the first two figures in column N.

Thus the $\log 516. = 2.7126$, $\log .00281 = -3.4487$, $\log 7400. = 3.8692$,
$\log 600. = 2.7782$, $\log 50. = 1.6990$, $\log 4.00 = .6021$.

EXERCISES

Find the logarithms of the following:

1. 3.	**2.** 303.	**3.** .024.
4. 347.	**5.** .0333.	**6.** 1.011.
7. .202.	**8.** .0029.	**9.** .0001.
10. .00299.	**11.** 68400.	**12.** .0201.

212. Interpolation. We find by the preceding rule that $\log 2440 = 3.3874$, while $\log 2450 = 3.3892$. If we seek the logarithm of a number between 2440 and 2450, say that of 2445, it would clearly be between 3.3874 and 3.3892. Since 2445 is just halfway between 2440 and 2450, we assume that its logarithm is halfway between the two logarithms. To find log 2445, then, we look up log 2440 and log 2450, take half (or .5) their difference, and add this to the log 2440. This gives

$$\log 2445 = 3.3874 + .5 \times .0018 = 3.3883.$$

If we had to find log 2442 we should take not half the difference but two tenths of the difference between the logarithms of 2440 and 2450, since 2442 is not halfway between them but two tenths of the way. This method is perfectly general, and we may always find the logarithm of a number of more than three figures by the following

RULE. *Annex to the proper characteristic the mantissa of the first three significant figures.*

Multiply the difference between this mantissa and the next larger mantissa in the table (called the tabular difference and denoted by D) by the remaining figures of the number preceded by a decimal point.

Add this product to the extreme right of the logarithm of the first three figures, rejecting all decimal places beyond the fourth.

In this process of interpolation we have assumed and used the principle that the increase of the logarithm is proportional to the increase of the number. This principle is not strictly true, though for numbers whose first significant figure is greater than 1 the error is so small as not to appear in the fourth place of the logarithm. For numbers whose first significant figure is less than 2 this error would often appear if we found the fourth place by interpolation. For this reason the table on pp. 248, 249 gives the logarithms of all such numbers exact to four figures, and in this part of the table we do not need to interpolate at all.

EXERCISES

Find the logarithms of the following:

1. 63.924.

Solution:

$$\begin{array}{rl} \log 63.9 = & 1.8055 \\ & \underline{2} \\ \log 63.924 = & 1.8057 \end{array} \qquad \begin{array}{r} \text{Tabular difference} = 7 \\ \underline{.24} \\ 28 \\ \underline{14} \\ 1.68 \end{array}$$

We add 2 to 1.8055 rather than 1, because 1.68 is nearer 2 than 1. In general we take the nearest integer.

2. 269.4. **3.** 1001. **4.** 62230.

5. 392.8. **6.** 9.365. **7.** 20060.

8. .4283. **9.** .3101. **10.** 9.999.

11. 82.93. **12.** .05273. **13.** 5.7828.

14. .003011. **15.** .002156. **16.** 3.1416.

17. 275.4×1.463.

Solution:

$$\begin{array}{rl} \log 275.4 = & 2.4399 \\ \log 1.463 = & \underline{0.1652} \\ \text{By Theorem II, § 209, } (\log 275.4 \times 1.463) = & 2.6051 \end{array} \qquad \begin{array}{rr} \log 275 = 2.4393 & D = 16 \\ \underline{6} & \underline{.4} \\ 2.4399 & 6.4 \end{array}$$

18. 374.8×1396. **19.** 1.46×237.2.

20. 469.1×63.92. **21.** $47320. \times .8994$.

22. $\dfrac{.03724}{38.46}$.

Solution:

$$\begin{array}{rl} \log .03724 = & 8.5710 - 10 \\ \log 38.46 = & \underline{1.5850 } \\ & 6.9860 - 10 \end{array} \qquad \begin{array}{rr} \log .0372 = 8.5705 - 10 & D = 12 \\ \underline{5} & \underline{.4} \\ 8.5710 - 10 & 4.8 \\ \log 38.46 = 1.5843 & D = 12 \\ \underline{7} & \underline{.6} \\ 1.5850 & 7.2 \end{array}$$

23. $\dfrac{3.467}{.2364}$. **24.** $\dfrac{.06792}{5.128}$.

213. Antilogarithms. We can now find the product or quotient of two numbers if we are able to find the number that corresponds to a given logarithm.

For this process we have the following

RULE. *If the mantissa is found exactly in the table, the first two figures of the corresponding number are found in the column N of the same row, while the third figure of the number is found at the top of the column in which the mantissa is found.*

Place the decimal point so that the rules in § 210 *are fulfilled.*

EXERCISES

Find the antilogarithms of the following:

1. 3.7419.

Solution: We find the mantissa .7419 in the row which has 55 in column N. The column in which .7419 is found has 2 at the top. Thus the significant figures of the antilogarithm are 552. Since the characteristic is 3, we must by the rule in § 210 have four figures to the left of the point. Thus the number sought is 5520.

2. 1.3874. **3.** 2.7050. **4.** .6785.

5. $\bar{2}.8414$.* **6.** 5.8831. **7.** $\bar{1}.5752$.

8. 9.9112 − 10. **9.** $\bar{3}.7251$. **10.** $\bar{5}.3997$.

If the mantissa of the given logarithm is between two mantissas in the table, we may find the antilogarithm by the following

RULE. *Write the number of three figures corresponding to the lesser of the two mantissas between which is the given mantissa.*

Subtract this mantissa from the given mantissa, and divide this number by the tabular difference to one decimal place.

Annex this figure to the three already found, and place the decimal point as the rules in § 210 *require.*

It should be kept in mind that we may always add and subtract any integer to a logarithm. This is useful in two cases:

First. When we wish to subtract a larger logarithm from a smaller;

Second. When we wish to divide a logarithm by an integer that is not exactly contained in the characteristic.

Both these processes are illustrated in exercise 2 (l) following.

EXERCISES

1. Find the antilogarithms of the following:

(a) 2.3469.

Solution: The mantissa 3469 is between 3464 and 3483. Hence $D = 19$.

The mantissa 3464 corresponds to 222. To find the fourth significant figure of the antilogarithm, divide 3469 − 3964 = 5 by $D = 19$. Since $5 \div 19 = 2.6$, we annex 3 to 222. Hence the antilogarithm = 222.3.

* We write $-2 + .8414$ in the form $\bar{2}.8414$ to save space and at the same time to recall the fact that the mantissa is positive.

(b) 4.3147. (c) 1.5271. (d) $\bar{1}.4216$.

(e) $\bar{1}.6423$. (f) $\bar{2}.8791$. (g) $.7214$.

2. Perform the following operations by logarithms.

(a) $\dfrac{1375 \times .06423}{76420}$.

Solution:

$$\begin{array}{lrl}
 & \log 1375 = & 3.1383 \\
 & \log .06423 = & \underline{8.8077 - 10} \\
\text{Adding (Theorem I, § 209),} & & 11.9460 - 10 \\
 & \log 76420 = & \underline{4.8832} \\
\text{Subtracting (Theorem III, § 209),} & \log \text{result} = & 7.0628 - 10 \\
 & \text{result} = & .001156.
\end{array}$$

(b) $(1\frac{7}{9})^3$. (c) $(\frac{9519}{8238})^6$. (d) $5871 \div 9308$.

(e) $(\frac{13}{11})^9$. (f) $(3\frac{27}{31})^{4.17}$. (g) $7065 \div 5401$.

(h) $8308 \times .0003769$. (i) $3410 \times .008763$.

(j) $\dfrac{8.371 \times 834.6}{7309}$. (k) $\dfrac{37.42 \times 11.21}{38.47}$.

(l) $\sqrt[3]{\dfrac{87 \times \sqrt{7194}}{98080000}}$.

Solution:

$$\begin{array}{lrl}
 & \log 87 = & 1.9395 \\
\text{By Theorem IV, § 209,} & \frac{1}{2}\log 7194 = & \underline{1.6285} \\
\text{Adding,} & = & 13.5670 - 10 \\
 & \log 98080000 = & \underline{7.9916} \\
\text{By Theorem III, § 209,} & 3) & \underline{25.5754 - 30} \\
 & \log \text{result} = & 8.5251 - 10 \\
 & \text{result} = & .03350.
\end{array}$$

Since in the subtraction in this problem we have to subtract 7 from 3, we add and subtract 10 to the minuend to avoid a negative logarithm. Since in the division by 3 we would have a remainder in dividing -10 by 3, we add and subtract 20 so that 3 may be exactly contained in 30, the negative part of the logarithm.

(m) $\sqrt[3]{7}$. (n) $\sqrt{.06}$. (o) $\sqrt[5]{(.043)^3}$.

(p) $\sqrt[11]{4}$. (q) $(.21)^{\frac{2}{3}}$. (r) $\frac{371}{834}\sqrt[3]{100}$.

(s) $\sqrt[3]{.03}$. (t) $\sqrt[5]{100}$. (u) $\sqrt{(1.563)^3}$.

(v) $\sqrt[3]{.00614}$. (w) $\sqrt{\frac{431}{788}}$. (x) $\sqrt[3]{0.9\sqrt[3]{\frac{29}{71}}}$.

(y) $\sqrt[3]{\frac{76}{93}} \cdot \sqrt{\frac{751}{518}}$. (z) $\sqrt[4]{.47\sqrt{\frac{19}{34}}}$.

214. Cologarithms. The logarithm of the reciprocal of a number is called its cologarithm. When a computation is to be made

in which several numbers occur in the denominator of a fraction, the subtraction of logarithms is conveniently avoided by the use of cologarithms. By our definition we have

$$\begin{aligned} \text{colog } 25 &= \log \tfrac{1}{25} = \log 1 - \log 25, \qquad \text{Theorem III, § 209} \\ \log 1 &= 10. \qquad -10 \\ \log 25 &= \underline{\ 1.3979\qquad\quad} \\ \text{colog } 25 &= \ 8.6021 - 10 \end{aligned}$$

Thus in dividing a number by 25 we may subtract the logarithm of 25, or what amounts to the same thing, add the logarithm of $\frac{1}{25}$, which is by definition the cologarithm of 25.

RULE. *The cologarithm of any number is found by subtracting its logarithm from 10 − 10.*

In the process of division subtracting the logarithm of a number and adding its cologarithm are equivalent operations.

EXERCISES

Compute, using cologarithms.

1. $\dfrac{8 \times 62.73 \times .052}{56 \times 8.793}$.

Solution:

$$\begin{aligned} \log 8 &= \ \ .9031 \\ \log 62.73 &= \ 1.7975 \\ \log .052 &= \ 8.7160 - 10 \\ \text{colog } 56 &= \ 8.2518 - 10 \\ \text{colog } 8.793 &= \underline{\ 9.0559 - 10} \\ \log \text{result} &= 27.7242 - 30 \\ \text{result} &= \ \ .005299 \end{aligned}$$

2. $\frac{109}{712}\sqrt{\frac{76}{93}}$.

3. $\sqrt{157^2 - 87^2}$.

HINT. $157^2 - 87^2 = (157 + 87)(157 - 87) = (244)(70)$.

4. $\sqrt{38.46^2 - 15.38^2}$.

5. $\dfrac{5086\,(.0008769)^3}{9802\,(.001984)^4}$.

6. $\sqrt{(27.5)^2 - (3.483)^2}$.

7. $\sqrt[3]{\dfrac{693 \times .04692}{.03841 \times (569.8)^2}}$.

8. $\sqrt{\dfrac{(58.96)^{\frac{1}{2}} \times 86.51}{.09263\ \sqrt[3]{50}}}$.

9. $\sqrt{\dfrac{(.058)^3 \times 421.6 \times 8^6}{\sqrt{50} \times .045 \cdot (200.1)^{\frac{1}{2}}}}$.

10. $\sqrt[3]{\dfrac{3.1416 \times (5.638)^2}{(75)^{\frac{2}{3}}}}$.

215. Change of base. We have seen that the logarithm of a number for the base 10 may be found to four decimal places in our tables. It is occasionally necessary to find the logarithm of a number for a base different from 10. For the sake of generality, we assume that the logarithms of all numbers for a base b are computed. We seek a means of finding the logarithm of any number, as x, for the base c; that is, we seek to express $\log_c x$ in terms of logarithms for the base b.

Suppose $\log_c x = z$, that is, $c^z = x$.

Take the logarithm of this equation for the base b, and we have

$$\log_b c^z = z \log_b c = \log_b x.$$

Then $$z = \frac{\log_b x}{\log_b c}.$$

If we let $$M = \log_b c,$$

we have $$z = \frac{\log_b x}{M}. \qquad (1)$$

This number M does not depend on the particular number x, but only on the two bases. From (1) we see that we can find the logarithm of any number for the base c by dividing its logarithm for the base b by M. The number M is called the **modulus** of the new system with respect to the original one.

RULE. *To find the logarithm of a number for a new base c, divide the common logarithm by the modulus of the system whose base is c.*

EXERCISES

Find:

1. $\log_3 21$.

Solution: $\log_3 21 = \dfrac{\log_{10} 21}{\log_{10} 3} = \dfrac{1.3222}{.4771} = 2.771.$

2. $\log_5 6$. **3.** $\log_2 15$. **4.** $\log_{15} 2$.

5. $\log_8 167$. **6.** $\log_{18} 237$. **7.** $\log_{2.16} 1.41$.

N.	0	1	2	3	4	5	6	7	8	9
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
101	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082
102	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124
103	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166
104	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208
105	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249
106	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290
107	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330
108	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370
109	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410
110	0414	0418	0422	0426	0430	0434	0438	0441	0445	0449
111	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488
112	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527
113	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565
114	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603
115	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641
116	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678
117	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715
118	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752
119	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824
121	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860
122	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896
123	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931
124	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966
125	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000
126	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035
127	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069
128	1072	1075	1079	1082	1086	1089	1093	1096	1099	1103
129	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136
130	1139	1143	1146	1149	1153	1156	1159	1163	1166	1169
131	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202
132	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235
133	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268
134	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300
135	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332
136	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364
137	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396
138	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427
139	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458
140	1461	1464	1467	1471	1474	1477	1480	1483	1486	1489
141	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520
142	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550
143	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581
144	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611
145	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641
146	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670
147	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700
148	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729
149	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787
151	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816
152	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844
153	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872
154	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901
155	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928
156	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956
157	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984
158	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011
159	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066
161	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092
162	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119
163	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146
164	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172
165	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198
166	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225
167	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251
168	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276
169	2279	2281	2284	2287	2289	2292	2294	2297	2299	2302
170	2304	2307	2310	2312	2315	2317	2320	2322	2325	2327
171	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353
172	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378
173	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403
174	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428
175	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453
176	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477
177	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502
178	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526
179	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550
180	2553	2555	2558	2560	2562	2565	2567	2570	2572	2574
181	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598
182	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622
183	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646
184	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669
185	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693
186	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716
187	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739
188	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762
189	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785
190	2788	2790	2792	2794	2797	2799	2801	2804	2806	2808
191	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831
192	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853
193	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876
194	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898
195	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920
196	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942
197	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964
198	2967	2969	2971	2973	2975	2978	2980	2982	2984	2986
199	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008
200	3010	3012	3015	3017	3019	3021	3023	3025	3028	3030
N.	0	1	2	3	4	5	6	7	8	9

N.	0	1	2	3	4	5	6	7	8	9
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

N.	0	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

216. Exponential equations. Equations in which the variable occurs only in the exponents may often be solved by the use of tables of logarithms if one keeps in mind the fact that

$$\log a^x = x \log a.$$

EXERCISES

Solve the following:

1. $10^{x-1} = 4$.

Solution: Taking the logarithm of both sides of the equation, we have

$$(x - 1) \log 10 = \log 4,$$

or since $\log 10 = 1$, $x = \log 4 + 1 = .6021 + 1 = 1.6021$.

2. $4^x \cdot 3^y = 8,$
$2^x \cdot 8^y = 9.$

Solution: Taking the logarithms of the equations, we have

$$x \log 4 + y \log 3 = \log 8,$$
$$x \log 2 + y \log 8 = \log 9,$$

or
$$x\, 2 \log 2 + y \log 3 = 3 \log 2, \qquad (1)$$
$$x \log 2 + y\, 3 \log 2 = 2 \log 3.$$

Eliminate x.

$$\begin{aligned} x\,2\log 2 + \quad y \log 3 &= 3 \log 2 \\ x\,2 \log 2 + y\,6 \log 2 &= 4 \log 3 \\ \hline y(\log 3 - 6 \log 2) &= 3 \log 2 - 4 \log 3 \end{aligned}$$

$$y = \frac{3 \log 2 - 4 \log 3}{\log 3 - 6 \log 2} = \frac{3 \times .3010 - 4 \times .4771}{.4771 - 6 \times .3010}$$

$$= \frac{.9030 - 1.9084}{.4771 - 1.8060} = \frac{-1.0034}{-1.3289} = \frac{1.0034}{1.3289}.$$

Perform this division by logarithms.

$$\begin{aligned} \log 1.0054 &= 10.0023 - 10 \\ \log 1.3289 &= \underline{\quad .1235 \quad} \\ \log y &= 9.8788 - 10 \\ y &= .7565. \end{aligned}$$

Substituting in (1),

$$\frac{x = 3 \log 2 - .7565 \log 3}{2 \log 2} = \frac{.9030 - .7565 \times .4771}{.6020}.$$

Compute $.7565 \times .4771$ by logarithms.

$$\begin{aligned} \log .7565 &= 9.8788 - 10 \\ \log .4771 &= \underline{9.6786 - 10} \\ \log \text{result} &= 19.5574 - 20 \\ \text{result} &= .3609. \end{aligned}$$

Hence
$$x = \frac{.9030 - .3609}{.6020} = \frac{.5421}{.6020}.$$

$$\begin{aligned} \log .5421 &= 19.7341 - 20 \\ \log .6020 &= \underline{9.7796 - 10} \\ \log x &= 9.9545 - 10 \\ x &= .9005. \end{aligned}$$

3. $6^x = 2.$ **4.** $4^y = 3.$ **5.** $7^{x+3} = 5.$

6. $3^{2x+1} = 5.$ **7.** $4^{x-1} = 5^{x+1}.$ **8.** $2^{2x+3} - 6^{x-1} = 0.$

9. $a^x \cdot b^y = m,$
$c^x \cdot d^y = n.$

10. $a^x \cdot b^y = m,$
$x + y = n.$

11. $2^x \cdot 2^y = 2^{22}$,
$x - y = 4$.

12. $a^{2x-3} \cdot a^{3y-2} = a^8$.
$3x + 2y = 17$.

13. $3^x \cdot 4^y = 15552$,
$4^x \cdot 5^y = 128000$.

14. $\sqrt[3]{a^{2x-1}} \cdot \sqrt[4]{a^{3y-1}} = a^8$,
$\sqrt[4]{b^{3x+5}} \cdot \sqrt[3]{b^{2y+1}} = b^{10}$.

217. Compound interest. If \$1500 is at the yearly interest of 3%, the total interest for a year is \$1500 $\cdot$ (0.03) = \$45. The total sum invested at the end of a year would be \$1545.

Let, in general, P represent a sum of money in dollars.

Let r represent a yearly rate of interest.

Then $P \cdot r$ represents the yearly interest on P, and

$$P + P \cdot r = P(r+1)$$

represents the total investment, principal and interest, at the end of a year.

Similarly, $P(r+1)r$ is the second year's interest, and

$$P(r+1)r + P(r+1) = P(r^2 + r + r + 1) = P(r+1)^2$$

is the total investment at the end of two years.

In general, $$A = P(r+1)^n \qquad (1)$$

is the total accumulation at the end of n years. If we know r, P, and n, we can by (1) find A. If we take the logarithm of both sides of the equation, we have

$$\log A = \log P + n \log(r+1),$$

or $$n = \frac{\log A - \log P}{\log(r+1)}. \qquad (2)$$

Hence if we know A, P, and r, we can find n.

If the interest is computed semiannually, we have as interest at the end of a half-year $P \cdot \frac{r}{2}$, while the entire sum would be $P\left(\frac{r}{2}+1\right)$. Reasoning as above, we find that if the interest is computed semiannually, the accumulation at the end of n years is

$$A = P\left(\frac{r}{2}+1\right)^{2n}. \qquad (3)$$

Similarly, $$n = \frac{\log A - \log P}{2\log\left(\frac{r}{2}+1\right)}. \qquad (4)$$

If the interest is computed k times a year, we have at the end of n years

$$A = P\left(\frac{r}{k} + 1\right)^{kn}, \tag{5}$$

or

$$n = \frac{\log A - \log P}{k \log\left(\frac{r}{k} + 1\right)}. \tag{6}$$

EXERCISES

In such exercises as the following, four-place tables are not sufficiently exact to obtain perfect accuracy. In general, the longer the term of years and the more frequent the compounding of interest, the greater the inaccuracy.

1. If \$1600 is placed at $3\frac{1}{2}\%$ interest computed semiannually for 13 years, to how much will it amount in that time?

Solution: By formula (3), $A = P\left(\frac{r}{2} + 1\right)^{2n}.$

$$P = 1600, \ r = .03\tfrac{1}{2}, \ n = 13.$$

Hence $A = 1600\left(\frac{.07}{4} + 1\right)^{26} = 1600\left(\frac{7}{400} + 1\right)^{26} = 1600\left(\frac{407}{400}\right)^{26}.$

$$\begin{aligned}
\log 1600 &= 3.2041 \\
26 \log 407 &= \underline{67.8496} \\
& \ 71.0537 \\
26 \log 400 &= \underline{67.6546} \\
\log A &= 3.3991 \\
A &= \$2507.
\end{aligned}
\qquad\qquad
\begin{array}{r}
\log 407 = 2.6096 \\
\underline{26} \\
156576 \\
\underline{52192} \\
67.8496 \\
\log 400 = 2.6021 \\
\underline{26} \\
156126 \\
\underline{52042} \\
67.6546
\end{array}$$

2. After how long will \$600 at 6% computed annually amount to \$1000?

Solution: By formula (2) we have

$$n = \frac{\log A - \log P}{\log(r + 1)}.$$

$$A = 1000, \ P = 600, \ r = .06.$$

$$n = \frac{\log 1000 - \log 600}{\log 1.06} = \frac{3 - 2.7782}{.0253} = \frac{.2318}{.0253} = 9.12 \text{ years.}$$

$$.12 \text{ year} = .12 \cdot 12 = 1.44 \text{ months.}$$

$$.44 \text{ month} = .44 \cdot 30 = 13.20 \text{ days.}$$

Thus $n = 9$ years 1 month 13.2 days.

In the following exercises the interest is computed annually unless the contrary is stated.

3. To what will $3750 amount in 20 years if left at 5% interest?

4. To what sum will $25,300 amount in 10 years if left at 5% interest computed semiannually?

5. To what does $1000 amount in 10 years if left at 6% interest computed (1) annually, (2) semiannually, (3) quarterly, (4) monthly?

6. A sum of money is left 22 years at 4% and amounts to $17,000. How much was originally put at interest?

7. What sum of money left at $4\frac{1}{2}$% for 30 years amounts to $30,000?

8. What sum of money left 10 years at $4\frac{1}{2}$% amounts to the same sum as $8549 left 7 years at 5%?

9. If a man left a certain sum 11 years at 4%, it would amount to $97 less than if he had left the same sum 9 years at 5%. What was the sum?

10. Which yields more, a sum left 10 years at 4% or 4 years at 10%? What is the difference for $1000?

11. Two sums of money, $25,795 in all, are left 20 years at $4\frac{3}{4}$%. The difference in the sums to which they amount is $14,660. What were the sums?

12. At what per cent interest must $15,000 be left in order to amount to $60,000 in 32 years?

13. At what per cent must $3333 be left so that in 24 years it will amount to $10,000?

14. Two sums of which the second is double the first but is left at 2% less interest amount in $36\frac{1}{2}$ years to equal sums. At what per cent interest was each left?

15. In how many years will a sum double if left at 5% interest?

16. In how many years will a sum double if left at 6% interest computed semiannually?

17. In how many years will a sum amount to ten times itself if left at 4% interest?

18. In how many years will $17,000 left at $4\frac{1}{2}$% interest amount to the same as $7000 left at $5\frac{1}{2}$% for 20 years?

19. On July 1, 1850, the sum of $1000 was left at $4\frac{1}{2}$% interest. When paid back it amounted to $2222. When did this occur?

20. Prove formulas (1), (3), and (5) by complete induction.

CHAPTER XXI

CONTINUED FRACTIONS

218. Definitions. A fraction in the form

$$a+\cfrac{b}{c+\cfrac{d}{e+\cfrac{f}{g+\cdots,}}}$$

where $a, b, \cdots, g, \cdots$ are real numbers, is called a **continued fraction.** We shall consider only those continued fractions in which the numerators b, d, f, etc., are equal to unity and in which the letters represent integers, as for example

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cdots,}}} \qquad \text{written } a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots.$$

When the number of quotients $a_2, a_3, a_4, \cdots$ is finite the fraction is said to be **terminating.** When the fraction is not terminating it is **infinite.** We shall see that the character of the numbers represented by terminating fractions differs widely from that of the numbers represented by infinite continued fractions. We shall find, in fact, that any root of a linear equation in one variable, i.e. any rational number, may be represented by a terminating continued fraction, and conversely; furthermore, that any real irrational root of a quadratic equation may be represented by the simplest type of infinite continued fractions, and conversely.

219. Terminating continued fractions. If we have a terminating continued fraction, where $a_1, a_2, \cdots$ are integers, it is evident that by reducing to its simplest form we obtain a rational number. The converse is also true, as we can prove in the following

THEOREM. *Any rational number may be expressed as a terminating fraction.*

Let $\frac{a}{b}$ represent a rational number. Divide a by b, and let a_1 be the quotient and c (which must be less than b) the remainder. Then (§ 26)

$$\frac{a}{b} = a_1 + \frac{c}{b} = a_1 + \frac{1}{\frac{b}{c}}.$$

Divide b by c, letting a_2 be the quotient and d (which must be less than c) the remainder. Then

$$\frac{a}{b} = a_1 + \frac{1}{a_2 + \frac{d}{c}}.$$

Continuing this process, the maximum limit of the remainders in the successive divisions becomes smaller as we go on, until finally the remainder is zero. Hence the fraction is terminating. It is noted that the successive *quotients* are the denominators in the continued fraction.

EXERCISES

1. Convert the following into continued fractions

(a) $\frac{77}{247}$.

Solution:

```
247|77|0
     77|247|3
        231
         16|77|4
            64
            13|16|1
               13
                3|13|4
                  12
                   1|3|3
                     3
                     0
```

The continued fraction is

$$\frac{77}{247} = \frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3}.$$

(b) $\frac{13}{97}$. (c) $\frac{61}{437}$. (d) $\frac{204}{457}$.
(e) $\frac{115}{151}$. (f) $\frac{408}{985}$. (g) $\frac{613}{2363}$.
(h) $\frac{1500}{2149}$. (i) $\frac{505}{192}$. (j) $\frac{2627}{1567}$.

2. **Express the following continued fractions as integers.**

(a) $\frac{1}{2+}\,\frac{1}{3}$. (b) $\frac{1}{4+}\,\frac{1}{7}$. (c) $\frac{1}{a+}\,\frac{1}{b}$.

(d) $\frac{1}{x+}\,\frac{1}{x}$. (e) $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}$. (f) $\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}$.

(g) $\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2}$. (h) $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}$.

220. Convergents. The value obtained by taking only the first $n-1$ quotients in a continued fraction is called the nth **convergent** of the fraction.

Thus in the fraction

$$1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{6}$$

1 is the first convergent,

$$1+\frac{1}{2}=\frac{3}{2} \text{ is the second convergent,}$$

$$1+\cfrac{1}{2+\cfrac{1}{3}}=1+\frac{3}{7}=\frac{10}{7} \text{ is the third convergent, etc.}$$

When there is no whole number preceding the fractional part of the continued fraction the first convergent is zero. Thus in

$$\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{5}$$

$\frac{1}{2}$ is called the second convergent.

In the continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\,\frac{1}{a_5+}\cdots$$

let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \cdots$ represent the successive convergents expressed as rational fractions.

Then for the first convergent we have

$$a_1=\frac{p_1}{q_1}, \text{ or } p_1=a_1,\ q_1=1.$$

For the second convergent we have

$$a_1 + \frac{1}{a_2} = \frac{a_2 a_1 + 1}{a_2} = \frac{p_2}{q_2}, \text{ or } p_2 = a_2 a_1 + 1 = a_2 p_1 + 1,$$
$$q_2 = a_2 = a_2 q_1.$$

For the third convergent we have

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} = a_1 + \frac{a_3}{a_3 a_2 + 1} = \frac{a_3(a_2 a_1 + 1) + a_1}{a_3 a_2 + 1} = \frac{p_3}{q_3},$$

or
$$p_3 = a_3(a_2 a_1 + 1) + a_1 = a_3 p_2 + p_1,$$
$$q_3 = a_3 a_2 + 1 = a_3 q_2 + q_1.$$

This indicates that the form of the nth convergent is

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}. \tag{1}$$

This is in fact the case, as we proceed to show by complete induction.

We have already established form (1) for $n = 2$ and $n = 3$. We assume it for $n = m$, and will show that its validity for $n = m + 1$ follows. The $(m + 1)$th convergent differs from the mth only in the fact that $a_m + \dfrac{1}{a_{m+1}}$ appears in the continued fraction in place of a_m. In (1) replace n by m, and a_n by $a_m + \dfrac{1}{a_{m+1}}$, and we have

$$\begin{aligned}
\frac{p_{m+1}}{q_{m+1}} &= \frac{\left(a_m + \dfrac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \dfrac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}} \\
&= \frac{(a_{m+1} a_m + 1)\, p_{m-1} + a_{m+1} p_{m-2}}{(a_{m+1} a_m + 1)\, q_{m-1} + a_{m+1} q_{m-2}} \\
&= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-2}} \\
&= \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}},
\end{aligned}$$

which is form (1).

EXERCISES

1. **Express the following as continued fractions, and find the convergents.**

(a) $\frac{30}{41}$.

Solution: By the method already explained, we find that

$$\frac{30}{41}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2}.$$

Here $a_1=0$, $a_2=1$, $a_3=2$, $a_4=1$, $a_5=2$, $a_6=1$, $a_7=2$.

The first convergent is evidently 0, the second is 1, and the third is $\dfrac{1}{1+\dfrac{1}{2}}=\dfrac{2}{3}$.

The fourth convergent is $\dfrac{p_4}{q_4}=\dfrac{a_4p_3+p_2}{a_4q_3+q_2}=\dfrac{1\cdot 2+1}{1\cdot 3+1}=\dfrac{3}{4}$.

The fifth convergent is $\dfrac{p_5}{q_5}=\dfrac{a_5p_4+p_3}{a_5q_4+q_3}=\dfrac{2\cdot 3+2}{2\cdot 4+3}=\dfrac{8}{11}$.

The sixth convergent is $\dfrac{p_6}{q_6}=\dfrac{a_6p_5+p_4}{a_6q_5+q_4}=\dfrac{1\cdot 8+3}{1\cdot 11+4}=\dfrac{11}{15}$.

The seventh convergent is $\dfrac{p_7}{q_7}=\dfrac{a_7p_6+p_5}{a_7q_6+q_5}=\dfrac{2\cdot 11+8}{2\cdot 15+11}=\dfrac{30}{41}$.

(b) $\frac{65}{352}$. (c) $\frac{55}{89}$. (d) $\frac{86}{121}$. (e) $\frac{57}{161}$. (f) $\frac{100}{137}$.
(g) $\frac{75}{194}$. (h) $\frac{241}{1760}$. (i) $\frac{56}{457}$. (j) $\frac{900}{3361}$.

2. Find the value of the following by finding the successive convergents.

(a) $\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{2}$. (b) $\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{3}$.

(c) $\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{2}$. (d) $\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3}$.

(e) $\dfrac{1}{5+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{5}$. (f) $\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{5+}\,\dfrac{1}{5+}\,\dfrac{1}{3+}\,\dfrac{1}{1}$.

(g) $\dfrac{1}{(x-1)+}\,\dfrac{1}{x+}\,\dfrac{1}{(x+1)}$. (h) $\dfrac{1}{x+}\,\dfrac{1}{x+}\,\dfrac{1}{x}$.

221. Recurring continued fractions. We have seen that every terminating continued fraction represents a rational number, and conversely. We now discuss the character of the numbers represented by the simplest infinite continued fractions. A **recurring** continued fraction is one in which from a certain point on a group of denominators is repeated in the same order.

Thus
$$\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\cdots,$$
$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots$$
are recurring continued fractions if the denominators are assumed to repeat indefinitely as indicated.

That a repeating continued fraction actually represents a number we shall establish in § 223. Unless this fact is proven, one runs the risk of dealing with symbols which have no meaning. If for certain continued fractions the successive convergents increase without limit, or take on erratic values that approach no limit, it is important to discover the fact. All the fractions that we discuss actually represent numbers, as we shall see.

We shall consider only continued fractions in which every denominator has a positive sign.

THEOREM. *Every recurring continued fraction is the root of a quadratic equation.*

Let, for instance, $$x=\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\cdots.$$

Evidently the part of the fraction after the first denominator c may be represented by x, and we have thus virtually the terminating fraction
$$x=\frac{1}{a+}\frac{1}{b+}\frac{1}{c+x}.$$

The second convergent is $\frac{1}{a}$.

The third convergent is
$$\frac{b}{ab+1}=\frac{p_3}{q_3}.$$

The fourth convergent, or x, gives us
$$x=\frac{p_4}{q_4}=\frac{a_4p_3+p_2}{a_4q_3+q_2}=\frac{(c+x)b+1}{(c+x)(ab+1)+a}.$$

Simplifying, we get
$$(ab+1)x^2+[c(ab+1)+a-b]x-bc-1=0,$$

which is a quadratic equation whose root is x, the value of the continued fraction.

Since this equation has a negative number for its constant term it has one positive and one negative root. The continued fraction must represent the positive root, since we assume that the letters a, b, c represent positive integers. The quadratic equation whose root is a recurring continued fraction with positive denominators will always have one positive and one negative root. The equation will be quadratic, however, whatever the signs of the denominators may be.

The proof may be extended to the case where there are any number of recurring denominators or any number of denominators before the recurrence sets in. Since every real irrational root of a quadratic equation is a surd, our result is equivalent to the statement that every recurring continued fraction may be expressed as a surd.

EXERCISES

Of what quadratic equations are the following roots? Express the continued fraction as a surd.

1. $\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\cdots.$

Solution: Let $x = \frac{1}{2+}\frac{1}{3+x}$.

Then
$$x = \cfrac{1}{2+\cfrac{1}{3+x}} = \frac{3+x}{6+2x+1},$$

or
$$2x^2 + 6x - 3 = 0.$$

Solving this equation, we get

$$x_1 = \frac{-3+\sqrt{15}}{2} \text{ or } x_2 = \frac{-3-\sqrt{15}}{2}.$$

Since x_2 is negative, x_1 must be the surd that is represented by the continued fraction.

Thus
$$\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\cdots = \frac{-3+\sqrt{15}}{2}.$$

2. $\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots.$

3. $\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\cdots.$

4. $\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots$.

5. $\frac{1}{3+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\cdots$.

6. $2+\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\cdots$.

7. $3+\frac{1}{3+}\frac{1}{4+}\frac{1}{5+}\cdots$.

HINT. If $x=2+\frac{1}{2+}\frac{1}{1+}\cdots$,

then $x-2=\frac{1}{2+}\frac{1}{1+}\cdots$

and $x=2+\frac{1}{2+}\frac{1}{1+(x-2)}$.

8. $1+\frac{1}{3+}\frac{1}{4+}\frac{1}{3+}\frac{1}{4+}\cdots$.

9. $1+\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots$.

10. $\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots$.

222. Expression of a surd as a recurring continued fraction. This is the converse of the problem discussed in the last section, and shows that recurring continued fractions and quadratic equations are related in the same intimate way that terminating fractions and rational numbers (i.e. the roots of linear equations) are connected. We seek to express an irrational number, as, for instance, $\sqrt{2}$, as a continued fraction. This we may do as follows.

Since 1 is the largest integer in $\sqrt{2}$ we may write

$$\sqrt{2}=1+(\sqrt{2}-1)=1+\frac{(\sqrt{2}-1)}{1}.$$

Rationalizing the numerator, we have

$$\sqrt{2}=1+\frac{1}{\sqrt{2}+1}.$$

Since 2 is the largest integer in $\sqrt{2}+1$ we have

$$\sqrt{2}=1+\frac{1}{2+(\sqrt{2}-1)}=1+\frac{1}{2+\frac{(\sqrt{2}-1)}{1}}.$$

Rationalizing the numerator $\sqrt{2}-1$, we have

$$\sqrt{2}=1+\frac{1}{2+}\frac{1}{\sqrt{2}+1}=1+\frac{1}{2+}\frac{1}{2+(\sqrt{2}-1)}.$$

By continuing this process we continually get the denominator 2. Thus

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\cdots.$$

This process consists of the successive application of two operations, and affords the

RULE. *Express the surd as the sum of two numbers the first of which is the largest integer that it contains.*

Rationalize the numerator of the fraction whose numerator is the second of these numbers. Repeat these operations until a recurrence of denominators is observed.

This process may be applied to any surd, and a continued fraction which is recurring will always be obtained. We shall content ourselves with a statement of this fact without proof.

If the surd is of the form $a - \sqrt{b}$, a continued fraction may be derived for $+\sqrt{b}$ and its sign changed. Since the real roots of any quadratic equation $x^2 + 2a_1x + a_2 = 0$ are surds of the form $a \pm \sqrt{b}$, where a and b are integers, it appears that the roots of any such equation may be expressed as recurring continued fractions. It can be shown that the real roots of the general quadratic equation $a_0x^2 + a_1x + a_2 = 0$ may also be so expressed.

EXERCISES

1. Express the following surds as recurrent continued fractions.

(a) $2 + \sqrt{3}$.

Solution:

$$2 + \sqrt{3} = 3 + (\sqrt{3} - 1) = 3 + \frac{\sqrt{3} - 1}{1} \tag{1}$$

$$= 3 + \frac{3 - 1}{\sqrt{3} + 1} = 3 + \frac{2}{\sqrt{3} + 1}$$

$$= 3 + \frac{1}{\dfrac{\sqrt{3}+1}{2}} = 3 + \frac{1}{1 + \dfrac{\sqrt{3}+1}{2} - 1} = 3 + \frac{1}{1 + \dfrac{\sqrt{3}-1}{2}}$$

$$= 3 + \frac{1}{1+}\,\frac{3 - 1}{2(\sqrt{3}+1)} = 3 + \frac{1}{1+}\,\frac{1}{\dfrac{\sqrt{3}+1}{1}} = 3 + \frac{1}{1+}\,\frac{1}{2 + (\sqrt{3} - 1)}$$

But since $\sqrt{3}-1$ is the same number that we have in (1), this fraction repeats from this point on, and we have

$$2+\sqrt{3}=3+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$$

(b) $\sqrt{5}$. (c) $\sqrt{17}$. (d) $\sqrt{65}$. (e) $\sqrt{47}$.

(f) $\sqrt{14}$. (g) $\sqrt{23}$. (h) $\sqrt{34}$. (i) $\sqrt{19}$.

(j) $\sqrt{62}$. (k) $\sqrt{79}$. (l) $\sqrt{98}$. (m) $\sqrt{88}$.

(n) $\sqrt{22}$. (o) $\sqrt{45}$. (p) $\sqrt{59}$. (q) $\sqrt{101}$.

(r) $7+\sqrt{11}$. (s) $8-\sqrt{3}$. (t) $3-\sqrt{23}$.

2. Express as a continued fraction the roots of the following equations.

(a) $x^2-7x-3=0$. (b) $x^2+2x-6=0$.

(c) $x^2+3x-8=0$. (d) $x^2-4x-4=0$.

223. Properties of convergents. The law of formation of convergents given in § 220 is valid whether the continued fraction is terminating or infinite. We should expect that in the case of an infinite fraction the successive convergents would give us an increasingly close approximation to the value of the fraction. This is indeed the fact, as we shall see.

THEOREM. *The difference between the nth and $(n+1)$st convergents is* $\dfrac{(-1)^{n+1}}{q_n q_{n+1}}$.

We prove this theorem by complete induction.

Let the continued fraction be

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdot$$

Then the first and second convergents are respectively

$$\frac{p_1}{q_1}=a_1 \text{ and } \frac{p_2}{q_2}=\frac{a_2a_1+1}{a_2}.$$

Then $$\frac{p_2}{q_2}-\frac{p_1}{q_1}=\frac{(a_2a_1+1)}{a_2}-a_1=\frac{1}{a_2}.$$

Since $$q_1 = 1,\ q_2 = a_2,$$

we have $$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_n q_{n+1}} \text{ for } n = 1.$$

We assume that the theorem holds for $n = m$, that is,

$$\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} = \frac{p_m q_{m+1} - q_m p_{m+1}}{q_m q_{m+1}} = \frac{(-1)^{m+1}}{q_m q_{m+1}}. \tag{1}$$

We must prove that it holds for $n = m + 1$.

Now since $\dfrac{p_{m+2}}{q_{m+2}} - \dfrac{p_{m+1}}{q_{m+1}} = \dfrac{p_{m+1}q_{m+2} - p_{m+2}q_{m+1}}{q_{m+1}q_{m+2}}$,

our theorem reduces to proving that the numerator

$$p_{m+1}q_{m+2} - p_{m+2}q_{m+1} = (-1)^{m+2}. \tag{2}$$

In the left-hand member of (2) set

$$a_{m+2}q_{m+1} + q_m = q_{m+2}, \qquad (1),\ §\ 220$$

and $$a_{m+2}p_{m+1} + p_m = p_{m+2}.$$

Then
$$\begin{aligned} & p_{m+1}(a_{m+2}q_{m+1} + q_m) - (a_{m+2}p_{m+1} + p_m)q_{m+1} \\ &= p_{m+1}q_m - p_m q_{m+1} = -(p_m q_{m+1} - p_{m+1}q_m) \\ &= -(-1)^{m+1} = (-1)^{m+2}. \qquad \text{by (1)} \end{aligned}$$

COROLLARY I. *The difference between the successive convergents of a continued fraction with positive denominators approaches zero as a limit.*

Since $q_n = a_n q_{n-1} + q_{n-2}$, evidently q_n increases without limit when n is increased, since to obtain q_n we add together positive numbers neither one of which can vanish.

Thus we can find a value of n large enough so that $\dfrac{1}{q_n}$, and hence $\dfrac{1}{q_n q_{n+1}}$, will be smaller than any assigned number, which is another way of stating that as n increases $\dfrac{1}{q_n q_{n+1}}$ approaches zero as a limit.

COROLLARY II. *The even convergents decrease, while the odd convergents increase, as n increases.*

We must show that

$$\frac{p_{m+2}}{q_{m+2}} - \frac{p_m}{q_m}$$

is negative or positive according as m is even or odd. Adding and subtracting $\frac{p_{m+1}}{q_{m+1}}$, we have

$$\begin{aligned}\frac{p_{m+2}}{q_{m+2}} - \frac{p_m}{q_m} &= \left(\frac{p_{m+2}}{q_{m+2}} - \frac{p_{m+1}}{q_{m+1}}\right) + \left(\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m}\right)\\ &= \frac{(-1)^{m+2}}{q_{m+2}q_{m+1}} + \frac{(-1)^{m+1}}{q_{m+1}q_m}.\end{aligned}$$

By Corollary I, the denominator of the first fraction exceeds that of the second. Hence when m is odd the sum in the last member of the equation is positive, and when m is even the sum is negative.

We now see that any recurring fraction of the type considered in § 221 actually represents a number in the sense of § 74. We have seen that the successive odd convergents continually increase, while the even convergents continually decrease, until the difference between a pair of them is very small. Such sequences of numbers we have seen (§ 73 ff.) define real numbers.

224. Limit of error. We are now in a position to state a maximum value for the error made in taking any convergent of a continued fraction for the fraction itself.

THEOREM. *The maximum limit of error in taking the nth convergent for the continued fraction is less than* $\frac{1}{q_n q_{n+1}}$.

Since by the theorem of the last section the value of the fraction is between any pair of consecutive convergents, it must differ from either of these convergents by less than they differ from each other, that is, by less than $\frac{1}{q_n q_{n+1}}$.

EXERCISES

Find a convergent that differs by less than .001 from each of the following:

1. $\sqrt{6}$.

Solution:

$$\sqrt{6} = 2 + (\sqrt{6} - 2) = 2 + \frac{6-4}{\sqrt{6}+2} = 2 + \cfrac{1}{\cfrac{\sqrt{6}+2}{2}}$$

$$= 2 + \frac{1}{2+}\left(\frac{\sqrt{6}+2}{2} - 2\right) = 2 + \frac{1}{2+}\frac{\sqrt{6}-2}{2}$$

$$= 2 + \frac{1}{2+}\frac{6-4}{2(\sqrt{6}+2)} = 2 + \frac{1}{2+}\frac{1}{\sqrt{6}+2} = 2 + \frac{1}{2+}\frac{1}{4+}(\sqrt{6}-2).$$

Since the last surd repeats the one in the first equation we have

$$\sqrt{6} = 2 + \frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4+}\cdots.$$

$$\frac{p_1}{q_1} = \frac{2}{1}; \quad \frac{p_2}{q_2} = \frac{5}{2}; \quad \frac{p_3}{q_3} = \frac{4\cdot 5+2}{4\cdot 2+1} = \frac{22}{9};$$

$$\frac{p_4}{q_4} = \frac{2\cdot 22+5}{2\cdot 9+2} = \frac{49}{20}; \quad \frac{p_5}{q_5} = \frac{4\cdot 49+22}{4\cdot 20+9} = \frac{218}{89}.$$

Since $$\frac{1}{q_4 q_5} = \frac{1}{20\cdot 89} = \frac{1}{1780} < .001,$$

we see by § 224 that $\frac{49}{20}$ satisfies the condition of the problem.

2. $\sqrt{7}$. 3. $\sqrt{46}$. 4. $\sqrt{3}$. 5. $\sqrt{19}$.

6. $\sqrt{35}$. 7. $\sqrt{32}$. 8. $\sqrt{61}$. 9. $\sqrt{55}$.

10. $3+\sqrt{2}$. 11. $\sqrt{99}$. 12. $\sqrt{41}$. 13. $\sqrt{13}$.

14. The number π has the value 3.14159. Find by the method of continued fractions a series of convergents the last of which differs from this value by less than .0001.

CHAPTER XXII

INEQUALITIES

225. General theorems. We say that a is greater than b when $a-b$ is positive. If $a-b$ is negative, then a is less than b. Thus any positive number or zero is greater than any negative number. As we distinguished between identities and equations of condition in § 53, so in this discussion we observe that some statements of inequality are true for any real value of the letters, while others hold for particular values only. The former class may be called **unconditional** inequalities, the latter **conditional**.

Thus $a^2 > -1$ is true for any real value of a and is unconditional, while $x - 1 > 2$ only when x is greater than 3 and is consequently conditional.

The two inequalities $a > b$, $c > d$ are said to have the same sense. Similarly, $a < b$, $c < d$ have the same sense. The inequalities $a > b$, $c < d$ have a different sense.

THEOREM I. *Any positive number may be added to, subtracted from, or multiplied by both numbers of an inequality without affecting the sense of the inequality.*

Let $a > b$, that is, let $a - b = k$, where k is a positive number. If m is a positive number, evidently

$$a \pm m - (b \pm m) = k,$$

or

$$a \pm m > b \pm m.$$

Similarly,

$$ma - mb = mk,$$

or

$$ma > mb.$$

The other statements of the theorem are proved similarly.

COROLLARY. *Terms may be transposed from one side of an equality to the other as in the case of equations.*

Let $a > b + c$.

Subtract c from both sides of the inequality and we obtain by Theorem I

$$a - c > b.$$

THEOREM II. *If the signs of both sides of an inequality are changed, the sense of the inequality must be reversed, that is, the $>$ sign must be changed to $<$, or conversely.*

Let $a > b$, that is, let $a - b = k$, where k is a positive number.

Then $$-a + b = -k,$$

or $$(-a) - (-b) = -k,$$

that is, by definition, $$-a < -b.$$

EXERCISES

Prove that the following identities are true for all real positive values of the letters.

1. $a^2 + b^2 > 2ab$.

Solution: $(a - b)^2$ is always positive.

Thus $$a^2 - 2ab + b^2 = a^2 + b^2 - 2ab \text{ is positive.}$$

That is, $$a^2 + b^2 > 2ab.$$

2. $3(a^3 + b^3) > a^2b + ab^2$.

3. $a^2 + b^2 + c^2 > ab + ac + bc$.

4. $(b + c)(c + a)(a + b) > 8abc$.

5. $(a + b + c)(a^2 + b^2 + c^2) > 9abc$.

6. $b^2c^2 + c^2a^2 + a^2b^2 > abc(a + b + c)$.

7. $3(a^3 + b^3 + c^3) > (a + b + c)(ab + bc + ca)$.

8. $\sqrt{(x + x_1)^2 + (y + y_1)^2} < \sqrt{x^2 + y^2} + \sqrt{x_1^2 + y_1^2}$.

9. If $a^2 + b^2 = 1$, $x^2 + y^2 = 1$, prove that $ax + by < 1$.

10. $(a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2 > ab + bc + ca$.

11. Show that the sum of any positive number (except 1) and its reciprocal is greater than 2.

12. Prove that the arithmetical mean of two unequal positive numbers always exceeds their geometrical mean.

226. Conditional linear inequalities. If we wish to find the values of x for which

$$ax + b < c, \tag{1}$$

where a, b, and c are numbers and a is positive, we may find such values by carrying out a process similar to that of solving a linear equation in one variable.

By the corollary, § 225, we have from (1)

$$ax < c - b.$$

By Theorem I, § 225, $x < \frac{c-b}{a}.$

227. Conditional quadratic inequalities. We have already shown in § 116 that the quadratic expression $ax^2 + bx + c$ is positive or negative, when the equation

$$ax^2 + bx + c = 0 \tag{1}$$

has imaginary or equal roots, according as a is positive or negative. If the equation has distinct real roots, the expression is positive or negative for values between those roots according as a is negative or positive. This we may express in tabular form as follows, for all values of x excepting the roots of (1), for which of course the expression vanishes.

a	$b^2 - 4ac$	$ax^2 + bx + c$
+	− or 0	Always +
−	− or 0	Always −
+	+	− for x between roots, + for other values
−	+	+ for x between roots, − for other values

This enables us to answer immediately questions like the following:

Example. For what values of x is $-2x^2+x > -3$? By the corollary, § 225, this is equivalent to the question, For what value of x is $-2x^2 + x + 3 > 0$?

Here $b^2 - 4ac = 1 + 24 = 25$ is positive. The roots of the equation $-2x^2 + x + 3 = 0$ are $x = -1$, $x = \frac{3}{2}$. Thus by our table this expression is positive for all values of x between -1 and $\frac{3}{2}$.

EXERCISES

For what values of x are the following inequalities valid?

1. $2x - 3 > 0$.
2. $4x - 7 > 1$.
3. $-x - 1 > 7$.
4. $-3x + 8 < 3$.
5. $\frac{9x}{3} + \frac{4}{7} > 1$.
6. $-\frac{2x}{3} + \frac{3}{4} < \frac{4}{5}$.
7. $.12x + .3 < 1.3$.
8. $3 - 4x > 2$.
9. $3 < 5x - 2$.
10. $.8 < \frac{4x}{11} - 1$.
11. $x^2 - 8x + 22 > 6$.
12. $x^2 + 3x - 2 > 1$.
13. $2x^2 - 3x > 5$.
14. $-3x^2 - 4x > 8$.
15. $2x^2 - 4x < -2$.
16. $3x^2 - 9x > -6$.
17. $-3x^2 + 2x < 2$.
18. $-x^2 + 6x > 9$.
19. $5x^2 - 8x < 1$.
20. $x^2 < x - 1$.
21. $3x^2 > 3x - 3$.
22. $3x > 2x^2 - 4$.

CHAPTER XXIII

VARIATION

228. General principles. The number x is said to vary **directly** as the number y when the ratio of x to y is constant. This we symbolize by

$$x \propto y, \text{ or } \frac{x}{y} = k, \tag{1}$$

where k is a constant.

Thus if a man walks at a uniform speed, the distance that he goes varies directly as the time. If the length of the altitude of a triangle is given, the area of the triangle varies directly as the base. The volume of a sphere varies directly as the cube of its radius.

The number x is said to vary **inversely** as the number y when x varies directly as the reciprocal of y. Thus x varies inversely as y when

$$x \propto \frac{1}{y}, \text{ or } \frac{x}{\frac{1}{y}} = xy = k, \tag{2}$$

where k is a constant. Thus the speed of a horse might vary inversely as the weight of his load. The length of time to do a piece of work might vary inversely as the number of laborers employed.

The intensity of a light varies inversely as the square of the distance of the light from the point of observation. If l represents the intensity of light and d the distance of the light from the point of observation, we have

$$l \propto \frac{1}{d^2}, \text{ or } \frac{l}{\frac{1}{d^2}} = ld^2 = k, \tag{3}$$

where k is a constant.

The number x is said to vary **jointly** as y and z when it varies directly as the product of y and z. Thus x varies jointly as x and z when

$$x \propto yz, \text{ or } \frac{x}{yz} = k, \tag{4}$$

where k is a constant.

Thus a man's wages might vary jointly as the number of days and the number of hours per day that he worked.

The number x is said to vary directly as y and inversely as z when it varies directly with $\frac{y}{z}$. Thus the force of the attraction of gravitation between two bodies varies directly as their masses and inversely as the squares of their distances. If m represents the masses of two bodies, d their distance, and G the force of their attraction due to gravity, then

$$G \propto \frac{m}{d^2}, \text{ or } \frac{Gd^2}{m} = k. \tag{5}$$

EXERCISES

1. If a varies inversely as the square of b, and if $a = 2$ when $b = 3$, what is the value of a when b is 18?

Solution: By (3), $ab^2 = k.$

We can determine k by substituting $a = 2$, $b = 3$.

$$2 \cdot 9 = k.$$
$$18 = k.$$

Then $a \cdot (18)^2 = 18,$

or $a = \frac{1}{18}.$

2. The volume of a sphere varies as the cube of its radius. A sphere of radius 1 has a volume 4.19. What is the volume of a sphere of radius 3?

Solution: Let V represent the volume and r the radius of the sphere. Then by (1),

$$\frac{V}{r^3} = k.$$

Determine k by substituting, $\frac{4.19}{1} = k.$

$$k = 4.19.$$

Then $\frac{V}{(3)^3} = \frac{V}{27} = 4.19$

$$V = 113.13.$$

3. If $xy \propto x + y$, and $x = 1$ when $y = 1$, find x when $y = 8$.

4. The area of a circle varies as the square of the radius. If a circle of radius 1 has an area 3.14, find the area of a circle whose radius is 21.

5. Find the volume of a sphere whose radius is .2.

HINT. See exercise 2.

6. The volume of a circular cylinder varies jointly with the altitude and the square of the radius of the base. A cylinder whose altitude and radius are each 1 has a volume of 3.14. Find the volume of a cylinder whose altitude is 15 and whose radius is 3.

7. The weight of a body of a given material varies directly with its volume. If a sphere of radius 1 inch weighs $\frac{3}{4}$ of a pound, how much would a ball of the same material weigh whose radius is 16 inches?

8. The distance fallen by an object starting from rest varies as the square of the time of falling. If a body falls 16 feet in 1 second, how far will it fall in 6 seconds?

9. A body falls from the top to the bottom of a cliff in $3\frac{1}{2}$ seconds. How high is the cliff?

10. A triangle varies in area jointly as its base and altitude. The area of a triangle whose base and altitude are each 1 is $\frac{1}{2}$. What is the area of a triangle whose base is 16 and altitude 7?

11. If 6 men do a piece of work in 10 days, how long will it take 5 men to do it?

12. If 3 men working 8 hours a day can finish a piece of work in 10 days, how many days will 8 men require if they work 9 hours a day?

13. An object is 30 feet from a light. To what point must it be moved in order to receive (a) half as much light, (b) three times as much light?

14. The weights of objects near the earth vary inversely as the squares of their distances from the center of the earth. The radius of the earth is about 4000 miles. If an object weighs 150 pounds on the surface of the earth, how much would it weigh 5000 miles distant from the surface?

CHAPTER XXIV

PROBABILITY

229. Illustration. If a bag contains 3 white balls and 4 black balls, and 1 ball is taken out at random, what is the chance that the ball drawn will be white?

This question we may answer as follows: There are 7 balls in the bag and we are as likely to get one as another. Thus a ball may be drawn in 7 different ways. Of these 7 possible ways 3 will produce a white ball. Thus the chance that the ball drawn will be white is 3 to 7, or $\frac{3}{7}$. The chance that a black ball will be drawn is $\frac{4}{7}$.

230. General statement. It is plain that we may generalize this illustration as follows: If an event may happen in p ways and fail in q ways, each way being equally probable, the chance or probability that it will happen in one of the p ways is

$$\frac{p}{p+q}. \tag{1}$$

The chance that it will fail is

$$\frac{q}{p+q}. \tag{2}$$

The sum of the chances of the event's happening and failing is 1, as we observe by adding (1) and (2).

The odds in favor of the event are the ratio of the chance of happening to the chance of failure. In this case the odds in favor are

$$\frac{p}{q}. \tag{3}$$

The odds against the event are $\frac{q}{p}$.

EXERCISES

1. If the chance of an event's happening is $\frac{1}{10}$, what are the odds in its favor?

Solution: By (1), $$\frac{p}{p+q}=\frac{1}{10}.$$

Hence $$10p = p + q,$$
or $$9p = q,$$
or $$\frac{p}{q}=\frac{1}{9},$$ which by (3) are the odds in favor.

2. From a pack of 52 cards 3 are missing. What is the chance that they are all of one suit?

Solution: The number of combinations of 52 cards taken 3 at a time is $c_{52,3}=\frac{52\cdot 51\cdot 50}{1\cdot 2\cdot 3}$. This represents $p+q$. The number of combinations of the 13 cards of any one suit taken 3 at a time is $c_{13,3}=\frac{13\cdot 12\cdot 11}{1\cdot 2\cdot 3}$. This represents p.

Thus $$\frac{p}{p+q}=\frac{\dfrac{13\cdot 12\cdot 11}{1\cdot 2\cdot 3}}{\dfrac{52\cdot 51\cdot 50}{1\cdot 2\cdot 3}}=\frac{13\cdot 12\cdot 11}{52\cdot 51\cdot 50}=\frac{11}{17\cdot 25}=\frac{11}{425}.$$

3. What is the chance of throwing one and only one 6 in a single throw of two dice?

Solution: There are 36 possible ways for the two dice to fall. This represents $p+q$. Since a throw of two sixes is excluded there are 5 throws in which each die would be a 6, that is, 10 in all in which a 6 appears. This represents p.

Thus $$\frac{p}{p+q}=\frac{10}{36}=\frac{5}{18}.$$

4. A bag contains 8 white and 12 black balls. What is the chance that a ball drawn shall be (a) white, (b) black?

5. A bag contains 4 red, 8 black, and 12 white balls. What is the chance that a ball drawn shall be (a) red, (b) white, (c) not black?

6. In the previous problem, if 3 balls are drawn, what is the chance that (a) all are black, (b) 2 red and 1 white?

7. What is the chance of throwing neither a 3 nor a 4 in a single throw of one die?

8. What is the chance in drawing a card from a pack that it be (a) an ace, (b) a diamond, (c) a face card?

9. Three cards are missing from a pack. What is the chance that they are (a) of one color, (b) face cards, (c) aces?

10. A coin is tossed twice. What is the chance that heads will fall once?

11. The chance that an event will happen is $\frac{2}{7}$. What are the odds in its favor?

12. The odds against the occurrence of an event are $\frac{4}{5}$. What is the chance of its happening?

13. What is the chance of throwing 10 with a single throw of two dice?

14. A squad of 10 men stand in line. What is the chance that A and B are next each other?

15. What is the chance that in a game of whist a player has 6 trumps?

16. What is the chance that in a game of whist a player holds 4 aces?

CHAPTER XXV

SCALES OF NOTATION

231. General statement. The ordinary numbers with which we are acquainted are expressed by means of powers of 10. Thus

$$263 = 2 \cdot 10^2 + 6 \cdot 10^1 + 3.$$

This is the common **scale of notation**, and 10 is called the **radix** of the scale.

In a similar manner a number might be expressed in any scale with any radix other than 10. If we take 6 as the radix, we shall have as a number in this scale, for instance,

$$563 = 5 \cdot 6^2 + 6 \cdot 6 + 3.$$

In this scale we need only 0 and five digits to express every positive integer.

In general, if r is the radix of a scale of notation, any positive integer N will be denoted in this scale as follows:

$$N = a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n. \qquad (1)$$

THEOREM. *Any positive integer may be expressed in a scale of notation of radix r.*

Suppose we have a positive integer N. Let r^n be the highest power of r that is contained in N. Then

$$N = a_0 r^n + N_1,$$

where N_1 is less than r^n. Suppose that on dividing N_1 by r^{n-1} we obtain

$$N_1 = a_1 r^{n-1} + N_2,$$

where N_2 is less than r^{n-1}.

Then $$N = a_0 r^n + a_1 r^{n-1} + N_2.$$

Proceeding in this manner we obtain finally

$$N = a_0 r^n + a_1 r^{n-1} + \cdots + a_n,$$

where the a's are positive integers less than r, or perhaps zeros.

One observes that the symbol 10 indicates the radix in any system. In this general scale we need a 0 and $r-1$ digits to express every possible number.

232. Fundamental operations. In the four fundamental operations in the common scale we carry and borrow 10 in computing. In computing in a scale of radix 6, for instance, we should carry and borrow 6. If the radix were r, we should carry or borrow r.

Thus let $r=6$. Then $4+5=1\cdot 6+3=13$. Similarly, $5\cdot 3=2\cdot 6+3=23$. This is precisely analogous to our computation in the common scale, where, for instance, we would have $9+8=1\cdot 10+7=17$, or $6\cdot 7=4\cdot 10+2=42$.

EXERCISES

Perform the following operations.

1. $2361+4253+2140$; $r=7$.

$$\begin{array}{r} 2361 \\ 4253 \\ \underline{2140} \\ 12114 \end{array}$$

In this process, since $3+1=4$ and is less than the radix, there is nothing to carry. The next column gives $6+5+4=15=2\cdot 7+1$, hence we write down 1 and carry 2. The next column gives $3+2+1+2=8=1\cdot 7+1$, hence we write down 1 and carry 1. Finally we get $2+4+2+1=9=1\cdot 7+2$, hence we write 12.

2. $4602-3714$; $r=8$.

$$\begin{array}{r} 4602 \\ \underline{3714} \\ 666 \end{array}$$

Since we cannot take 4 from 2 we borrow one from the next place. Since the radix is 8 this amounts to 8 units in the first place. We then subtract 4 from $8+2$, which leaves 6. In borrowing 1 from 0 that digit is really reduced to 7 and the preceding digit to 5; then subtracting 1 from 7 we get 6. Since we cannot take 7 from 5 we borrow 8 again and take 7 from $5+8=13$, which leaves 6. Since 1 has been borrowed from the 4 we see the subtraction is complete since $3-3=0$.

3. $4321\cdot 432$; $r=5$.

$$\begin{array}{r} 4321 \\ \underline{432} \\ 14142 \\ 24013 \\ \underline{33334} \\ 4143222 \end{array}$$

In multiplying by 2 we have nothing to carry until we multiply 3 by 2. This gives $6=1\cdot 5+1$. Hence we put down 1 and carry 1 to the product of 2 and 4. The addition of the partial products is carried out as in exercise 1.

4. $32130 \div 43$; $r = 6$.

$$\begin{array}{r|l|l} 43 & 32130 & 430 \\ & 300 & \\ \hline & \ \ 213 & \\ & \ \ 213 & \\ \hline & \ \ \ \ 00 & \end{array}$$

In making an estimate for the first figure in the quotient we divide 32 by 4, keeping in mind that for this purpose $32 = 3 \cdot 6 + 2$. Thus 4 is contained in 20 just 5 times, but since our entire divisor is 43 we take 4 as the first figure in the quotient. The multiplications are of course performed as in exercise 3, excepting that here 6 is the radix.

5. $4361 + 2635 + 5542$; $r = 7$.

6. $5344 - 3456$; $r = 7$.

7. $2340 \cdot 4101$; $r = 5$.

8. $6435 \cdot 35$; $r = 7$.

9. $2003455 \div 403$; $r = 6$.

10. $344032 \div 321$; $r = 5$.

11. $534401 - 443524$; $r = 6$.

12. $425 + 254 + 542 + 452$; $r = 6$.

233. Change of scale. If we have a number in the scale of radix r, we may find the expression for that number in the common scale by writing the number in form (1), § 231, and carrying out the indicated operations.

Example. Convert 4635, where $r = 7$, into the ordinary scale.

$$\begin{aligned} 4635 &= 4 \cdot 7^3 + 6 \cdot 7^2 + 3 \cdot 7 + 5 \\ &= 4 \cdot 343 + 6 \cdot 49 + 3 \cdot 7 + 5 \\ &= 1692. \end{aligned}$$

If we have a number in the common scale, we may express it in the scale with radix r as follows: If the number is N, we have to determine the integers $a_0, a_1, \cdots, a_n$ in the expression

$$N = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n. \qquad (1)$$

Divide (1) by r. We have

$$\frac{N}{r} = a_0 r^{n-1} + a_1 r^{n-2} + a_{n-1} + \frac{a_n}{r} = N' + \frac{a_n}{r};$$

that is, the remainder a_n of this division is the last digit in the expression desired.

Divide N' by r and we obtain

$$\frac{N'}{r} = N'' = a_0 r^{n-2} + a_1 r^{n-3} + \cdots + \frac{a_{n-1}}{r};$$

that is, the remainder from this division is the next to the last digit in the desired expression. Proceeding in this way we obtain all the digits $a_n, a_{n-1}, \cdots, a_1, a_0$.

Example. Express 37496 in the scale with radix 7.

7) 37496
7) 5356 remainder 4
7) 765 remainder 1
7) 109 remainder 2
7) 15 remainder 4
7) 2 remainder 1
0 remainder 2

The number in scale $r = 7$ is 214214.

To change a number from any scale r_1 to any other scale r_2, we may first change the number to the scale of 10 and then by the process just given to the scale r_2. The process indicated in the preceding example may be employed directly to change from any scale to any other, provided the division is carried out in the scale in which the number is given. One of these methods may be used to check the other.

Example. Change 34503 from scale $r = 6$ to one in which $r = 9$.

$$34503 = 3 \cdot 6^4 + 4 \cdot 6^3 + 5 \cdot 6^2 + 3 = 4935 \text{ in scale of } 10.$$

9) 4935
9) 548 remainder 3
9) 60 remainder 8
9) 6 remainder 6
0 remainder 6

Thus 34503 in scale of 6 becomes 6683 in scale of 9.

Check: 9) 34503
9) 2312 remainder 3
9) 140 remainder 8
9) 10 remainder 6
0 remainder 6

In carrying out this division it must be kept in mind that the dividends are in scale of 6, while the remainders are to be in scale of 9.

234. Fractions. In the ordinary notation we express fractional numbers by digits following the decimal point. This notation may also be used in a scale with any radix.

Thus the expression .5421 stands for

$$\frac{5}{10}+\frac{4}{10^2}+\frac{2}{10^3}+\frac{1}{10^4}$$

in the common scale.

In the scale with radix r it stands for

$$\frac{5}{r}+\frac{4}{r^2}+\frac{2}{r^3}+\frac{1}{r^4}.$$

The process of changing the scale for fractions is performed in accordance with the same principles as are employed in the change of scale for integers. The following examples suffice to illustrate it.

EXAMPLE 1. Express .5421 in the scale of 6 as a decimal fraction.

$$\begin{aligned} .5421 &= \frac{5}{6}+\frac{4}{6^2}+\frac{2}{6^3}+\frac{1}{6^4} \\ &= \frac{5\cdot 6^3+4\cdot 6^2+2\cdot 6+1}{6^4}=\frac{1237}{1296}=.9545\cdots. \end{aligned}$$

EXAMPLE 2. Express .439 as a fraction for radix 6.

Let $\qquad .439=\dfrac{a}{6}+\dfrac{b}{6^2}+\dfrac{c}{6^3}+\dfrac{d}{6^4}+\cdots.$

Multiplying by 6, $\qquad 2.634=a+\dfrac{b}{6}+\dfrac{c}{6^2}+\dfrac{d}{6^4}+\cdots.$

Thus $a=2$ and we have $.634=\dfrac{b}{6}+\dfrac{c}{6^2}+\dfrac{d}{6^3}+\cdots.$

Multiplying by 6, $\qquad 3.804=b+\dfrac{c}{6}+\dfrac{d}{6^2}+\cdots.$

Thus $b=3$ and we have $.804=\dfrac{c}{6}+\dfrac{d}{6^2}+\cdots.$

Multiplying by 6, $\qquad 4.824=c+\dfrac{d}{6}+\cdots.$

Thus $c=4$ and we have $.824=\dfrac{d}{6}+\cdots.$

Multiplying by 6, $\qquad 4.944=d+\cdots.$

Thus $\qquad d=4.$

The fraction in scale of radix 6 is then $.2344\cdots$.

EXERCISES

1. Express the following as decimal fractions.

(a) $.374$; $r = 8.$ (b) $.4352$; $r = 6.$

(c) $.2231$; $r = 4.$ (d) $.2001$; $r = 3.$

2. Express the decimal fraction $.296$ as a radix fraction for $r = 5.$

3. Express the decimal fraction $.3405$ as a radix fraction for $r = 6.$

4. Express $\frac{34}{128}$ as a radix fraction for $r = 4.$

5. Express $\frac{151}{625}$ as a radix fraction for $r = 5.$

6. In what scale is 42 expressed as 1120?

Solution: We seek r where

$$r^3 + r^2 + 2r = 42.$$

This is equivalent to finding a positive integral root of the equation

$$r^3 + r^2 + 2r - 42 = 0.$$

By synthetic division,

$$\begin{array}{rrrr|l} 1 + & 1 + & 2 & - 42 & \underline{2} \\ & + 2 + & 6 & + 16 & \\ \hline & + 3 + & 8 & - 26 & \end{array}$$

$$\begin{array}{rrrr|l} 1 + & 1 + & 2 & - 42 & \underline{3} \\ & + 3 + & 12 & + 42 & \\ \hline & + 4 + & 14 & 0 & \end{array}$$

Thus 3 is the value sought.

Check: $3^3 + 3^2 + 2 \cdot 3 = 27 + 9 + 6 = 42.$

7. In what scale is 2704 denoted by 20304?

8. In what scale is 256 denoted by 10000?

9. In what scale is .1664 denoted by .0404?

10. Show that 1331 is a perfect cube.

235. Duodecimals. We may apply some of the foregoing processes to mensuration. If we take one foot as a unit and the radix as 12, we may express distances in the so-called duodecimal notation. Thus 2 ft. 6 in. is represented in the duodecimal scale by 2.6. Since in a scale of radix r we need $r - 1$ symbols, we will let $10 = t$ and $11 = e$. Thus 21 ft. 10 in. would be expressed in duodecimal notation as $19.t$. We may now find areas and volumes in this notation much more readily than by the usual method of converting all distances to inches.

EXAMPLE. Multiply 8 ft. 3 in. by 3 ft. 10 in. We multiply $8 \cdot 3$ by $3.t$ in the duodecimal scale. To convert the result to square feet and square inches we must keep in mind that $27.76 = 2 \cdot 12 + 7 + \frac{7}{12} + \frac{6}{144} = 31$ sq. ft. 90 sq. in., since 144 square inches equal one square foot.

$$\begin{array}{r} 8.3 \\ \underline{3.t} \\ 6t6 \\ \underline{209} \\ 27.76 \end{array}$$

This example suggests the following method of multiplying distances:

RULE. *Express the distances in duodecimal notation with the foot as a unit.*

Multiply in the scale for which $r = 12$.

In the product change the part on the left of the point from duodecimal to decimal scale.

Multiply the digit following the point by 12, and add to the last figure to obtain the square inches in the result.

EXERCISES

1. Multiply the following:

(a) 13 ft. 4 in. by 67 ft. 11 in.

Solution:

$$\begin{array}{r} 11.4 \\ \underline{57.e} \\ 1028 \\ 794 \\ \underline{568} \\ 635.68 \end{array} = 905 \text{ sq. ft. } 80 \text{ sq. in.}$$

(b) 10 ft. 6 in. by 12 ft. 2 in.

(c) 8 ft. 4 in. by 11 ft. 11 in.

(d) 23 ft. 6 in. by 47 ft. 8 in.

(e) 41 ft. 6 in. by 36 ft. 1 in.

2. What is the area of a room 16 ft. 2 in. by 10 ft. 3 in.?

3. What is the area of a walk 60 ft. 6 in. by 3 ft. 3 in.?

4. What is the area of a city lot 52 ft. 6 in. by 153 ft. 7 in.?